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Torus-Fibered Calabi-Yau threefolds with non-trivial fundamental group

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Abstract: We construct smooth Calabi-Yau threefolds Z, torus-fibered over a dP_9 base, with fundamental group $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$. To do this, the structure of rational elliptic surfaces is studied and it is shown that a restricted subset of such surfaces admit at least a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of automorphisms. One then constructs Calabi-Yau threefolds X as the fiber product of two such dP_9 surfaces, demonstrating that the involutions on the surfaces lift to a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of automorphisms on X. The threefolds Z are then obtained as the quotient $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. These Calabi-Yau spaces Z admit stable, holomorphic SU(4) vector bundles which, in conjunction with $\mathbb{Z}_2 \times \mathbb{Z}_2$ Wilson lines, lead to standard-like models of particle physics with naturally suppressed nucleon decay.

Keywords: M-Theory, Differential and Algebraic Geometry.

Contents

1. Introduction

In several papers [1, 2], Hořava and Witten showed that the strongly coupled heterotic string, M-theory compactified on an $S^1/\mathbb{Z}_2$ orbifold, is described by two end-of-the-world fixed ten-planes, each with an E_8 worldvolume supermultiplet, separated by the eleventh dimension. Furthermore, it was demonstrated in [3] that, when compactified on a Calabi-Yau threefold, this theory could, in the low momentum limit, describe realistic grand unified theories (GUTs) in four-dimensions. In the phenomenologically relevant case when the Calabi-Yau radius is much smaller than the size of the orbifold, the process of compactification proceeds through a five-dimensional effective theory, which was first computed in [4, 5]. This five-dimensional theory, called heterotic M-theory, is comprised of two end-of-the-world three branes, one our observable universe and the other a hidden sector, as well as a variable number of wrapped five-branes inside the five-dimensional geometrically warped bulk space. Thus, heterotic M-theory emerges as a fundamental paradigm for "brane universe" scenarios within the context of strongly coupled heterotic strings. The residual gauge supermultiplets on the end-of-the-world branes need no longer be E_8. Instead, non-trivial gauge instantons on the Calabi-Yau threefold can break these to smaller, and more physically relevant, subgroups. In [6, 7, 8], it was shown how to construct large classes of gauge instantons on elliptically fibered Calabi-Yau threefolds with trivial homotopy that break E_8 to phenomenologically interesting GUT theories with E_6, SO(10) and SU(5) gauge groups. These instantons, generically, have structure groups and connections that do not correspond to the "standard" SU(3) embedding.

A deficiency of this approach is that, apparently, it is impossible to break E_8 to the standard model gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, or even a standard model-like variation of this group, on a Calabi-Yau threefold with trivial fundamental group. The reason is straightforward. The most obvious mechanism to break the above GUT groups to the standard model is through extensions of the gauge instantons by a non-trivial flat bundle, that is, Wilson lines. However, these do not exist on a manifold where the fundamental group, π_1, is the identity. This deficiency was overcome in [9]–[13], where torus-fibered Calabi-Yau threefolds with $\pi_1 = \mathbb{Z}_2$ were constructed. Furthermore, in [10]–[13], stable gauge instantons with SU(5) structure group, leading to SU(5) GUT theories at low energy, were produced for the first time. When these instantons are extended by $\mathbb{Z}_2$ Wilson lines, the low energy gauge group is exactly $SU(3) \times SU(2) \times U(1)$. Therefore, within this context, a standard-like model of particle physics can be achieved.

As emphasized long ago by Witten [14], a potential problem of SU(5) models with $\mathbb{Z}_2$ Wilson lines is too rapid nucleon decay precipitated by dimension 4 operators. Although this need not be the case, there is no obvious symmetry that prevents it. One solution of this problem is to proceed not through SU(5) but, rather, through an SO(10) GUT group. When broken by Wilson lines to a standard-like model, the low energy gauge group often contains a $U(1)_{B-L}$ symmetry that eliminates the dimension 4 operators. It would seem worthwhile, therefore, to extend the results of [10]–[13] to SO(10) GUT theories broken by Wilson lines. However, since SO(10) is a larger group than SU(5), it is necessary to have a larger group of Wilson lines, at least $\mathbb{Z}_2 \times \mathbb{Z}_2$, to achieve a standard-like model. To do this, one must first construct torus-fibered Calabi-Yau threefolds with fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$ or larger. Once this is done, gauge instantons with structure group SU(4), which lead to unbroken SO(10) at low energy, must be produced on these manifolds. If this can be achieved, then Wilson lines will break this theory to standard-like models at low energy with naturally suppressed nucleon decay. In this paper, we accomplish the first of these tasks, namely, we construct torus-fibered Calabi-Yau threefolds with $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

To accomplish this, we proceed as follows. As stated above, our context will be Calabi-Yau threefolds, X, fibered over a base surface B'. As discussed elsewhere [8], the requirement that X be Calabi-Yau restricts the base surfaces to be either a Hirzebruch surface $\mathbb{F}_r$ where r is a non-negative integer, an Enriques surface, $\mathcal{E}$, a del Pezzo surface, dP_i for $i = 1, \ldots, 8$, or a rational elliptic surface dP_9. Of these, perhaps the most interesting one is dP_9, which is itself elliptically fibered over the base complex projective line $\mathbb{P}^1$. The fibered structure of dP_9 naturally admits involutions. Since we are interested in fundamental groups containing $\mathbb{Z}_2$ factors, it seems reasonable to choose $B' = dP_9$ for our construction.

With this in mind, in section 2 we discuss various properties of rational elliptic surfaces B that are essential in this paper. These include their structure as a fibration, the Kodaira classification of the fibers in the discriminant and a detailed construction of their Weierstrass model. We recall, in the process, that the complex dimension of the moduli space of generic rational elliptic surfaces is eight. Section 3 is devoted to a complete description of all allowed involutions τ_B on a rational elliptic surface. We show that these come in three different types. The first is of the form $\tau_B = t_\xi \circ (-1)_B$, where ξ is any section of B and $(-1)_B$ is the natural inversion on the fibers, while the second is $\tau_B = t_\xi$, where ξ intersecting any fiber is a point of order two. These two involution types share the property that their projection onto the base $\mathbb{P}^1$ is trivial. The third type of involution has the structure $\tau_B = t_\xi \circ \alpha_B$, where α_B is a zero section preserving involution mapping fibers to fibers and ξ satisfies an appropriate constraint. The projection of this third type of involution onto the base $\mathbb{P}^1$ is non-trivial. We indicate why the first type of involution is unlikely to induce a fixed point free involution on the full Calabi-Yau threefold X and, hence, do not consider it further. The second type of involution can induce a fixed point free action on X, but can only exist in restricted cases when a global section of order two is defined. Hence, this type of involution is of secondary importance and we discuss it only in the appropriate context. However, the third type of involution is of fundamental importance. These involutions, when ξ is not the zero section, can, indeed, lead to a freely acting involution on X. Furthermore, they occur on generic rational elliptic surfaces. Hence, the next few sections are dedicated to analyzing this third type of involution. This analysis follows the work of [10].

To begin with, we must show under what conditions an involution α_B exists. We examine this question in detail in section 4, proving that such involutions exist in a five-parameter subspace of the eight parameter moduli space of rational elliptic surfaces. This is accomplished by showing that any quotient space B/α_B must be the double cover of the product surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve $M = T \cup r$, where T and r have bidegree (1,4) and (1,0) respectively. The existence of such a branched surface then indicates the existence of an involution α_B on B. Counting the moduli space of these rational elliptic surfaces then reduces, in essence, to counting the moduli of the curves T and r.

Having identified the five parameter space of rational elliptic surfaces B that admit an involution α_B, we then discuss, in section 5, the allowed set of involutions of the form $\tau_B = t_\xi \circ \alpha_B$. This depends on the space of constrained sections ξ. For a generic surface B in this five-parameter space, such sections form a rank four lattice. Hence, there are many such involutions. However, we show that no two of these involutions commute and, therefore, that the maximal group of involutions on B is $\mathbb{Z}_2$. It was demonstrated in [10, 11, 12] that the homology ring of these surfaces does not contain a $\mathbb{Z}_2$ invariant fiber class, a pre-requisite for constructing standard-like models. Therefore, in the remainder of section 5, we further restrict the moduli space of rational elliptic surfaces with the intent of introducing such invariant fiber classes. This is accomplished by restricting the curve T of the branch locus in space Q to be of the form $T = t \cup s$, where curves t and s have bidegree (1,3) and (0,1) respectively. We show that the surfaces B constructed in this manner form a four parameter subspace of the five parameter space of surfaces admitting involutions α_B. The restriction on the moduli space of these surfaces also greatly reduces the number of constrained sections ξ. For a generic four parameter surface, these are reduced to the zero section, which is not of interest, plus three pairs of explicit sections. There are now only six involutions of the type $\tau = t_\xi \circ \alpha_B$ but, again, no two of these involutions commute and, hence, the maximal group of involutions remains $\mathbb{Z}_2$. However, we have achieved our goal. We show explicitly that the restriction to this four-parameter family of surfaces B introduces one extra fiber class which, as will be shown in [15], is invariant under the involution group. As demonstrated in [10, 11, 12], the existence of this class allows one to construct standard-like models within the context of SU(5) gauge instantons with $\mathbb{Z}_2$ Wilson lines.

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However, since we are interested in SU(4) gauge instantons with larger groups of involutions, we must place even further restrictions on the parameter space of rational elliptic surfaces B. This is accomplished in section 6. In this section, we further restrict the curve T of the branch locus in space Q to be of the form $T = t^{'} \cup i \cup s$, where curves $t^{'}$, i and s have bidegree $(1,2), (0,1)$ and $(0,1)$ respectively. We show that surfaces B constructed in this manner form a three-parameter subspace of the four-parameter space discussed above. The restriction on the moduli space of these surfaces further reduces the number of constrained sections ξ. Specifically, we show that, in the generic case, these are now reduced to a single section of order 2, which we call e_6, plus two pairs of explicit sections. The existence of the section of order two allows two involutions of the form $\tau_B = t_\xi \circ \alpha_B$ to commute. We find that the maximal group of commuting involutions is now enlarged to $\mathbb{Z}_2 \times \mathbb{Z}_2$, as we desired. However, a fundamental problem remains. In analogy with the situation in [10, 11, 12], it is essential to find a fiber class that is invariant under this enlarged group of involutions. Despite the fact that several new classes are introduced in this restricted moduli space, none of them is found to be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. To find such a class, we must make one further restriction.

Hence, for the remainder of section 6 we further restrict the curve T to be of the form $T = t^{''} \cup j \cup i \cup s$, where $t^{''}$ has bidegree $(1,1)$ and the remaining curves bidegree $(0,1)$. We show that the surfaces B constructed in this manner form a two-parameter subspace of the three-parameter moduli space above. As in previous cases, the number of restricted sections ξ is also reduced, now consisting of two different sections of order 2, the section e_6 discussed above plus a new section which we denote by e_4, and one remaining pair of explicit sections. In addition, we show that there is one extra section of order two, distinct from e_6 and e_4, that intersects this pair at a point and satisfies the constraint on ξ. This last section is not generic and appears because of the strong restriction on the surfaces B. The appearance of these extra sections of order two now allows three involutions of the form $\tau_B = t_\xi \circ \alpha_B$ to commute. We find that the maximal group of commuting involutions is now enlarged to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In addition, there is now an extra fiber class that is invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the involution group. This, as we will show elsewhere [16, 17], is sufficient to allow one to construct standard-like models in the SU(4), $\mathbb{Z}_2 \times \mathbb{Z}_2$ Wilson line context. Hence, although it is embedded in a larger involution group, it is actually this $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup that plays an active role.

In section 7, we construct Calabi-Yau threefolds X that are elliptically fibered over a base surface dP_9 and that admit a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of involutions. We begin by demonstrating that all Calabi-Yau threefolds elliptically fibered over $B^{'} = dP_9$ can be written as a fiber product $X = B \times_{\mathbb{P}^1} B^{'}$. We then compute the Chern classes of the tangent bundle of X explicitly, verifying that $c_1(T_X) = 0$ and that X admits a global, holomorphic three-form, two requisite properties for X to be a Calabi-Yau threefold. The factorization of X into a fiber product of rational elliptic surfaces allows us to lift involutions from these surfaces to the threefold. We show in section 7 that any Calabi-Yau threefold $X = B \times_{\mathbb{P}^1} B^{'}$, with both B and $B^{'}$ restricted to the two-parameter region of moduli space discussed in section 6, admits a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ as its maximally commuting set of involutions. Actually, one can lift the full $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ group of involutions to X, but only the $\mathbb{Z}_2 \times \mathbb{Z}_2$

subgroup acts freely. In addition, the invariant fiber classes on B and $B^{'}$ induce a $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant class on X. We now complete the program addressed in this paper by modding out the freely acting involution group on X to produce the threefold $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. It is then shown that Z has vanishing first Chern class and a global, holomorphic three-form and, hence, is a Calabi-Yau space. Unlike X, which is elliptically fibered, we show that Z admits no global section and, therefore, is only torus-fibered. Clearly, Z has the non-trivial homotopy group $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2$, as well as an extra class which descends from the invariant class on X. This was the goal of this paper.

This work is the first in a series of papers [15]–[18] which will construct standard-like models of particle physics with naturally suppressed nucleon decay based on structure group SU(4) gauge instantons and Wilson lines. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation properties of the homology ring of Z will be described in [15]. In [16] we will construct stable, holomorphic vector bundles with structure group SU(4) on Z and show that, with the addition of Wilson lines, these can be adjusted to give three family standard-like models. Finally, we will give a more detailed discussion of the underlying mathematics, much of which has been suppressed in the previous publications, in [17].

2. Rational elliptic surfaces B

In this section, we derive some of the basic properties of rational elliptic surfaces. We will give two different geometric descriptions of such surfaces and formulate their Weierstrass models.

Let $\mathbb{P}^2$ be a complex projective space of dimension two and denote a point in $\mathbb{P}^2$ by $(z_0 : z_1 : z_2)$ where z_i are projective coordinates. Specify by $l \cong \mathbb{P}^1$ a divisor of $\mathbb{P}^2$ given by a linear equation in the coordinates z_i. Then these coordinates can be considered as global sections in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l))$, where $\mathcal{O}_{\mathbb{P}^2}(l)$ is the line bundle on $\mathbb{P}^2$ with a unique section vanishing along the given line l. In fact, any curve $C \subset \mathbb{P}^2$ can be given as the zero set of a section of a line bundle on $\mathbb{P}^2$. Each curve characterizes a line bundle uniquely, which is denoted by $\mathcal{O}_{\mathbb{P}^2}(C)$. Since on $\mathbb{P}^2$ all line bundles with global holomorphic sections are isomorphic to $\mathcal{O}_{\mathbb{P}^2}(Nl)$ for positive N, it follows that any curve C can be specified by a homogeneous polynomial of degree N in the projective coordinates $(z_0 : z_1 : z_2)$. Here, we are interested in the case when C is a genus one Riemann surface, that is, a torus T^2. In this case, it is not difficult to show that $N = 3$. We conclude that a given torus T^2 can be "cut out" of $\mathbb{P}^2$ by specifying a unique homogeneous polynomial f of degree three in the complex projective coordinates.

With this in mind, we consider the product $\mathbb{P}^1 \times \mathbb{P}^2$ with the natural projections $\pi_i : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^i$ for $i = 1, 2$. Denote a point in $\mathbb{P}^1$ by $(t_0 : t_1)$, where $t_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ for $i = 0, 1$ are projective coordinates on $\mathbb{P}^1$, and a point in $\mathbb{P}^2$ by $(z_0 : z_1 : z_2)$, as above. Let

$$B \subset \mathbb{P}^1 \times \mathbb{P}^2 \tag{2.1}$$

be an effective divisor with associated line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(B)$. By definition, this line bundle admits a unique section f_t whose zero locus is precisely B. Extending the previous discussion, we see that f_t must be a homogeneous polynomial of degree M in the projective coordinates $(t_0 : t_1)$ and of degree N in the projective coordinates $(z_0 : z_1 : z_2)$, where M, N are positive integers. We define the space B by explicitly giving the section f_t, which is chosen to be the degree $(1, 3)$ polynomial

$$f_t = t_0 f_0 + t_1 f_1 \,, \tag{2.2}$$

where f_0, f_1 are two homogeneous polynomials of degree 3 in $(z_0 : z_1 : z_2)$.

What are the structure and properties of this specific space B ? To begin with, since B is a divisor in $\mathbb{P}^1 \times \mathbb{P}^2$

$$dim_{\mathbb{C}} B = 2 \,, \tag{2.3}$$

so B is a complex surface. Now note that B inherits two natural projections

$$\beta_i : B \to \mathbb{P}^i \tag{2.4}$$

for $i = 1, 2$ where

$$\beta_i = \pi_i|_B \,. \tag{2.5}$$

First consider $\beta_2 : B \to \mathbb{P}^2$. For any given point $z = (z_0 : z_1 : z_2) \in \mathbb{P}^2$, the fiber $\beta_2^{-1}(z)$ is given by the solution of the equation

$$t_0 f_0(z) + t_1 f_1(z) = 0 \,. \tag{2.6}$$

For a general point $z \in \mathbb{P}^2$, at least one of $f_0(z), f_1(z)$ does not vanish and, hence, the solution of (2.6) is a unique point in $\mathbb{P}^1$. However, as illustrated in figure 1, there are nine distinct points, call them ω_a with $a = 1, \ldots, 9$, where $f_0(\omega_a) = f_1(\omega_a) = 0$. It is clear that at each of these points the solution of equation (2.6) is the complete complex projective space $\mathbb{P}^1$. This leads to the first characterization of the surface B, namely, that B is $\mathbb{P}^2$ blown-up by $\mathbb{P}^1$ at each of the nine points satisfying $f_0 = f_1 = 0$. Specifically,

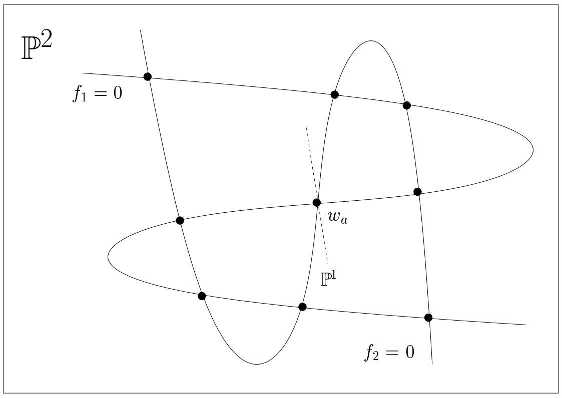

Figure 1: The curves $f_1 = 0$ and $f_2 = 0$ intersecting at nine points.

$$\beta_2 : B \to \mathbb{P}^2$$

is such that the pre-image $\beta^{-1}(\omega_a) \cong \mathbb{P}^1$ for $a = 1, \ldots, 9$. For further details on the blow-up of surfaces, we refer the reader to [19]. B is related to del Pezzo surfaces and, hence, is denoted by

$$B = d\mathbb{P}_9 \,. \tag{2.7}$$

Under a mild general position requirement, each subset of eight points ω_a determines the ninth. It follows that the surface B depends only on $2 \cdot 8 - dim\mathbb{P}GL(3, \mathbb{C}) = 8$ parameters, where $\mathbb{P}GL(3, \mathbb{C})$ is the automorphism group of $\mathbb{P}^2$. That is,

$$dim\mathcal{M}(B) = 8 \,, \tag{2.8}$$

where $\mathcal{M}(B)$ is the moduli space of generic surfaces B.

Now consider the remaining natural projection $\beta_1 : B \to \mathbb{P}^1$. For any given point $t = (t_0 : t_1) \in \mathbb{P}^1$, the fiber $\beta_1^{-1}(t)$ is given by the solution of the equation

$$t_0 f_0(z) + t_1 f_1(z) = 0 \,. \tag{2.9}$$

Since at least one of t_0, t_1 is non-zero, equation (2.9) is simply the zero locus of some homogenous polynomial of degree three in the projective coordinates $(z_0 : z_1 : z_2)$ of $\mathbb{P}^2$. It follows from the above discussion that this cuts out a locus $T^2 \subset \mathbb{P}^2$. That is

$$\beta_1^{-1}(t) \cong T^2 \,. \tag{2.10}$$

This leads to the second characterization of surface B. That is, with respect to the natural projection

$$\beta : B \to \mathbb{P}^1 \,, \tag{2.11}$$

where here and henceforth we denote β_1 by β, B is a torus fibration over $\mathbb{P}^1$.

In fact, B is an elliptic fibration, as we now demonstrate. First, note that every fiber $\beta^{-1}(t)$ intersects each of the nine blown-up $\mathbb{P}^1$ curves, since $f_0(\omega_a) = f_1(\omega_a) = 0$ for $a = 1, \dots, 9$ solves equation (2.9) for any value of t. It follows that each of these curves is a global section of the fibration (2.11). We denote these nine sections by e_i, $i = 1, \dots, 9$ where each $e_i \cong \mathbb{P}^1$. Now choose any one of these nine sections. For concreteness, we will always choose e_9. The intersection of e_9 with any fiber $\beta^{-1}(t) \cong T^2$ marks a point on T^2. This can be chosen as the identity element which turns T^2 into an abelian group. Since this choice is canonically given at each fiber, (2.11) becomes an elliptic fibration with zero section

$$e \equiv e_9 \,. \tag{2.12}$$

Note, that the space of global sections of (2.11) form a abelian group as well, with e as the identity element. Since B is elliptically fibered, and it is constructed as the blow-up of $\mathbb{P}^2$, it is referred to as a rational elliptic surface.

An important topological invariant of surface B is the Euler characteristic

$$\chi(B) = \sum_{i=0}^{4} (-1)^i rank(H_i(B, \mathbb{Z})) \,. \tag{2.13}$$

We first evaluate $\chi(B)$ using the characterization of B as a blow-up, $d\mathbb{P}_9$, of $\mathbb{P}^2$ at nine separated points. Since each blow-up adds a linearly independent class to $H_2(B, \mathbb{Z})$, we see

$$\chi(B) = \chi\left(\mathbb{P}^2\right) + 9 \,. \tag{2.14}$$

Then, using the fact that $\chi(\mathbb{P}^2) = 3$, we conclude that

$$\chi(B) = 12 \,. \tag{2.15}$$

Let us now evaluate $\chi(B)$ using the second characterization of B as the elliptic fibration (2.11). If B were to be a simple product $\mathbb{P}^1 \times T^2$ then, using the relation $\chi(\mathbb{P}^1 \times T^2) =$

$\chi(\mathbb{P}^1)\chi(T^2)$ and the fact that $\chi(T^2) = 0$, we would conclude that $\chi(B) = 0$, which is incorrect. We would arrive at the same conclusion if B was a T^2 fiber bundle over $\mathbb{P}^1$. It follows that there must be a set of points D in the base $\mathbb{P}^1$ of (2.11) such that, over every point $d \in D$, the fiber $\beta^{-1}(d)$ degenerates and is no longer a smooth T^2. Then $\chi(\beta^{-1}(d)) \neq 0$ and the above inconsistency can disappear. The set $D \subset \mathbb{P}^1$ is a divisor of $\mathbb{P}^1$ called the discriminant of the fibration.

To proceed, we need to specify $\chi(\beta^{-1}(d))$. There are many possible degenerations of the torus, each with a different topological structure and Euler characteristic. Such degenerations have been classified by Kodaira [20]. In this paper, we will be concerned with only two of these. The first, and the simplest, is called an I_1 fiber and is simply the degeneration of a single A-type one-cycle on T^2 to a point. This can be resolved topologically into a sphere by removing the singular point and adding two points at the ends of the tube and, hence,

$$\chi(I_1) = 1\,. \tag{2.16}$$

The second is called an I_2 fiber and consists of the union of two $\mathbb{P}^1$'s intersecting transversely in two distinct points. Therefore, its Euler characteristic is

$$\chi(I_2) = 2\,. \tag{2.17}$$

In our setup, the discriminant D will always be the union $D = D_1 \cup D_2$, where any point in D_i is associated with the fiber I_i for $i = 1, 2$. Then

$$\chi(B) = \chi(\mathbb{P}^1 \setminus D)\chi(T^2) + \sum_{i=1}^{2} \chi(D_i)\chi(I_i)\,. \tag{2.18}$$

Using (2.15), (2.16), (2.17) and the fact that $\chi(T^2) = 0$, this expression becomes

$$\chi(D_1) + 2\chi(D_2) = 12\,. \tag{2.19}$$

This has seven solutions of the form

$$\chi(D_1) = 2n\,, \qquad \chi(D_2) = m\,, n + m = 6\,, \tag{2.20}$$

where n and m are non-negative integers. The generic solution is given by $n = 6, m = 0$. In this case, $\chi(D_1) = 12$ and $D_2 = \emptyset$ indicating that D_1 consists of twelve isolated points. That is, in this case, the fibration (2.11) has twelve separated I_1 fibers. Now consider the solution given by $n = 5, m = 1$. Here, $\chi(D_1) = 10$ and $\chi(D_2) = 1$, indicating that D_1 consists of ten isolated points and D_2 has one point. Hence, the fibration (2.11) has ten separated I_1 fibers and one I_2 fiber. This second solution has a simple interpretation. Remember from the above discussion that $\chi(I_1) = 1$. Selecting any pair of separated I_1 fibers and pushing them together will produce a singular fiber f_{sing} with $\chi(f_{sing}) = 2$. However, there are two different Kodaira fibers with Euler characteristic two, I_2 and the cusp, which is of type II. In this paper, as we show in detail below, the degeneration is of the first type, that is, $f_{sing} = I_2$. This reduces the number of I_1 fibers from twelve to ten and introduces one I_2 fiber, precisely the second solution. Hence, the second solution is simply a special region in the moduli space of the generic solution of (2.20). Clearly, the same is true of the remaining five solutions. They are all obtained from the generic solution by combining pairs of I_1 fibers. For example, the $n = 0, m = 6$ solution has $D_1 = \emptyset$ and $\chi(D_2) = 6$, indicating that the fibration (2.11) has six separated I_2 fibers. This is obtained from the generic solution by choosing six pairs of I_1 fibers and moving in the moduli space until each pair merges into an I_2 fiber. These special points in moduli space will play a substantial role in the subsequent discussion.

It is reasonably clear from the previous discussion, and can be verified, that $H_2(B,\mathbb{Z})$ has a basis of classes given by $\beta_2^{-1}(l)$, which we will also denote as l, and $e_i, i = 1,\ldots,9$ with the intersection numbers

$$l^2 = 1\,, \qquad e_i \cdot e_j = -\delta_{ij}\,, \qquad l \cdot e_i = 0\,. \tag{2.21}$$

Note, for example, that the class of a generic fiber, denoted by f, can be written as

$$f = 3\ell - \sum_{i=1}^{9} e_i\,. \tag{2.22}$$

To see this, recall that, prior to blowing up, the fiber is given by a curve in the class $3l$. However, on the blow-up $B = d\mathbb{P}_9$ of $\mathbb{P}^2$, the pre-image of this class will contain the nine exceptional curves $e_i, i = 1,\ldots,9$. But, as we have demonstrated, these curves are sections of (2.11). Therefore, to obtain the pure fiber class one must subtract off each of the nine exceptional curves, yielding expression (2.22). The fibers are referred to as the proper transforms of the cubics in $\mathbb{P}^2$ passing through the points ω_a.

An important bundle on B is the canonical bundle $K_B = \wedge^2 T^* B$. Using the relation

$$K_B = \beta_2^* K_{\mathbb{P}^2} \otimes \sum_{i=1}^{9} \mathcal{O}_B(e_i)\,, \tag{2.23}$$

and the fact that $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3l)$, it follows that

$$K_B = \mathcal{O}_B\left(-3l + \sum_{i=1}^{9} e_i\right). \tag{2.24}$$

Hence, the canonical class, which we will also denote by K_B, is

$$K_B = -3l + \sum_{i=1}^{9} e_i = -f\,. \tag{2.25}$$

The elliptic fibration (2.11) can be given a Weierstrass representation as follows. First consider the projective bundle

$$p : P \to \mathbb{P}^1\,, \tag{2.26}$$

where

$$P = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3\right) \tag{2.27}$$

and

$$\mathcal{L} = N^*_{e/B} = \mathcal{O}_e(-e) \cong \mathcal{O}_{\mathbb{P}^1}(1) \tag{2.28}$$

is the conormal bundle of the zero section $e = e_9$. We have used the fact that the normal bundle of the divisor e is given by the restriction of $\mathcal{O}_B(e)$ to e, that is $N_{e/B} = \mathcal{O}_e(e)$. Using the intersection numbers (2.21), it follows that $N^*_{e/B} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Also note that $\mathcal{O}_P(1) \otimes p^*\mathcal{L}^m$ for any integer m is a line bundle over P. Then there are three canonical sections

$$x \in H^0\left(P, \mathcal{O}_P(1) \otimes p^*\mathcal{L}^2\right), \qquad y \in H^0\left(P, \mathcal{O}_P(1) \otimes p^*\mathcal{L}^3\right), \qquad z \in H^0\left(P, \mathcal{O}_P(1)\right). \tag{2.29}$$

Using the fact that $\mathcal{O}_P(1)|_{p^{-1}(b)\cong\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(l)$ for any $b \in \mathbb{P}^1$, we see that x, y, z restricted to any $\mathbb{P}^2$ fiber are coordinates of $\mathbb{P}^2$. In fact, they are precisely the $(z_0 : z_1 : z_2)$ projective coordinates defined previously. Now consider the line bundle $\mathcal{O}_P(3) \otimes p^*\mathcal{L}^6$ on P. The key point, which we will not prove here, is that the Weierstrass model of B, which we denote by W_B, satisfies

$$\mathcal{O}_P(W_B) = \mathcal{O}_P(3) \otimes p^*\mathcal{L}^6 . \tag{2.30}$$

Furthermore, there exist a map $\nu : B \to W_B$. Since for generic B the map ν is an isomorphism, we will, where further specification is not required, denote W_B simply as B. Then (2.30) implies that $B = d\mathbb{P}_9$ is an effective divisor in P. That is, B can be described as a submanifold in the total space of the projective bundle P over $\mathbb{P}^1$. Setting $\beta = p|_B$, we recover the fibration (2.11). Relation (2.30) is very useful in that it tells us that B can be cut out of P as the zero locus of some unique section

$$s \in H^0\left(P, \mathcal{O}_P(3) \otimes p^*\mathcal{L}^6\right). \tag{2.31}$$

What is the section s ? It follows from the explicit $\mathcal{O}_P(3)$ factor in (2.31) that is must be a homogeneous polynomial of degree three in the sections x, y, z. There are ten possible terms, which we group as

$$x^3,\, y^2z \qquad x^2y,\, xy^2,\, y^3 \qquad x^2z,\, xyz,\, xz^2,\, yz^2,\, z^3 .$$

Each term in the first group has the structure of (2.31) and is allowed. Now consider, for example, x^2y in the second group. Clearly

$$x^2y \in H^0\left(P, \mathcal{O}_P(3) \otimes p^*\mathcal{L}^7\right), \tag{2.32}$$

which must be multiplied by a coefficient which is a section of $p^*\mathcal{L}^{-1}$ if it is to match (2.31). But $p^*\mathcal{L}^{-1}$ admits no global holomorphic sections and, hence, x^2y as well as all other terms in the second group are disallowed. Similarly, each term in the third group has the form of (2.31) if it is multiplied by a coefficient which is a section of $p^*\mathcal{L}^m$ for $m > 0$. Since $p^*\mathcal{L}^m$ for $m > 0$ admit global sections, all terms in the third group are allowed. The construction of s is simplified somewhat be the realization that three of the terms in the last group, traditionally taken to be x^2z, xyz and yz^2, can be set to zero by the appropriate choice of x, y and z. It follows that

$$s = y^2z - x^3 - g_2xz^2 - g_3z^3 , \tag{2.33}$$

where

$$g_2 \in H^0(P, p^*\mathcal{O}_{\mathbb{P}^1}(4)), \qquad g_3 \in H^0(P, p^*\mathcal{O}_{\mathbb{P}^1}(6)) \tag{2.34}$$

are determined by the explicit choice of B. The surface W_B, that is, B is then given by the zero locus of s

$$y^2z = x^3 + g_2xz^2 + g_3z^3, \tag{2.35}$$

which is the Weierstrass representation of the fibration (2.11).

3. Involutions on B

In this section, we will give a complete classification of all possible involutions on rational elliptic surfaces and describe some of their features. Let $\tau_B : B \to B$ be any involution on B, that is, $\tau_B^2 = id$. Clearly τ_B is an automorphism of B and, hence,

$$\tau_B^* K_B \cong K_B. \tag{3.1}$$

However, it was shown previously in (2.25) that $K_B = \mathcal{O}_B(-f)$. If follows that τ_B preserves the fiber class f of the fibration $\beta : B \to \mathbb{P}^1$. Note that this does not mean that fibers need to be invariant under τ_B, only that τ_B maps fibers to fibers. We conclude that any involution $\tau_B : B \to B$ induces an involution $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ on the base satisfying $\tau_{\mathbb{P}^1}^2 = id$ and

$$\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \tau_B. \tag{3.2}$$

The possible involutions $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$ are easy to classify. One can show that for an appropriate choice of the projective coordinates $(t_0 : t_1)$ on $\mathbb{P}^1$, $\tau_{\mathbb{P}^1}$ acts either as

$$t_0 \to t_0, \qquad t_1 \to t_1, \tag{3.3}$$

that is, $\tau_{\mathbb{P}^1} = id$ or non-trivially as

$$t_0 \to t_0, \qquad t_1 \to -t_1, \tag{3.4}$$

which has two fixed points at

$$0 \equiv (1:0), \qquad \infty \equiv (0:1). \tag{3.5}$$

That 0 is a fixed point is obvious. To see that ∞ is indeed fixed, recall that projective coordinates are only determined up to multiplication by non-zero complex numbers. Therefore, after acting with $\tau_{\mathbb{P}^1}$ we are allowed to re-scale by -1 without changing the point. Hence, any involution on B acts on the base $\mathbb{P}^1$ either as the identity or as a non-trivial involution with two fixed points. Note also that a non-trivial involution on $\mathbb{P}^1$ is uniquely determined by its fixed points.

Let us first discuss the possible involutions τ_B on B whose induced involutions on $\mathbb{P}^1$ are $\tau_{\mathbb{P}^1} = id$. Such involutions τ_B clearly act along fibers, leaving each fiber stable, that is, invariant but not necessarily point-wise fixed.

One such involution, which we denote by $(-1)_B$, is immediately apparent and defined as follows. Recall from (2.11) and (2.12) that B is an elliptic fibration with the zero section chosen to be $e = e_9$. The zero section e marks a fixed origin, also called e, on each T^2 fiber. Choose one such fiber. Marking the origin automatically defines an abelian group on the points of T^2 with e as the zero element. We define $(-1)_B|_{T^2} : T^2 \to T^2$ by

$$(-1)_B|_{T^2}(a) = -a\,, \tag{3.6}$$

where a is any element of the abelian group T^2 and $-a$ its inverse. Clearly, $(-1)_B|^2_{T^2} = id$ and, hence, $(-1)_B|_{T^2}$ is an involution. Note that $(-1)_B|_{T^2}$ has four fixed points, the trivial fixed point e as well as three non-trivial fixed points e', e'' and e'''. $(-1)_B$ is defined as the mapping on B whose restriction to each fiber T^2 is $(-1)_B|_{T^2}$. Clearly $(-1)^2_B = id$ and, hence, $(-1)_B$ is an involution on B. Furthermore, since $(-1)_B$ leaves each fiber stable, it follows that the associated $\tau_{\mathbb{P}^1} = id$. Note that $(-1)_B$ is a natural involution arising on any elliptic fibration.

Are there other involutions τ_B such that $\tau_{\mathbb{P}^1} = id$? The answer is affirmative. Most of them arise in a specific way from $(-1)_B$. To construct them, we must first introduce the notion of a translational automorphism on B. Let $\Gamma(B)$ be the space of global sections of the fibration (2.11). $\Gamma(B)$ is non-empty since we know it contains $e_1, \ldots, e_9$ defined above. Furthermore, as mentioned previously, it has an abelian group structure. Let $\xi \in \Gamma(B)$. Then ξ marks a point, also called ξ, on each fiber T^2. Define the mapping $t_\xi|_{T^2} : T^2 \to T^2$ by

$$t_\xi|_{T^2}(a) = a + \xi\,, \tag{3.7}$$

where a is any point on T^2. Clearly, this map is a translation along the fiber T^2. Furthermore, unless $\xi = e, e', e''$ or e''', $t_\xi|_{T^2}$ is not an involution. Finally, we can define t_ξ on B as the map whose restriction to each fiber T^2 is $t_\xi|_{T^2}$. Clearly t_ξ is a translational automorphism of B. There will be such an automorphism associated with every section of the fibration (2.11). Now consider

$$\tau_B = t_\xi \circ (-1)_B\,. \tag{3.8}$$

Since

$$(-1)_B \circ t_\xi = t_{-\xi} \circ (-1)_B\,, \tag{3.9}$$

it is clear that $\tau_B^2 = id$ and, hence, τ_B is an involution on B. By construction, τ_B leaves every fiber stable and, thus, the associated involution is $\tau_{\mathbb{P}^1} = id$. On a generic surface, all involutions τ_B which act trivially as $\tau_{\mathbb{P}^1} = id$ on $\mathbb{P}^1$ are of the form (3.8). Note that only $(-1)_B$ preserves the zero section.

As we show later, for specific choices of B, there are other involutions whose induced action on $\mathbb{P}^1$ is trivial. They are of the form

$$\tau_B = t_\xi\,, \tag{3.10}$$

where $\xi \neq e$ is a section of B which intersects every fiber at a point which is invariant under $(-1)_B$, that is, at $e^{'}, e^{''}$ or $e^{'''}$. For generic B, such sections do not exist. However, they may exist as sections of restricted rational elliptic surfaces.

Let us now consider involutions τ_B on B whose induced involution on $\mathbb{P}^1$ is non-trivial, that is, of type (3.4). Here, we simply present the result which classifies all such involutions, referring the reader to [10] for an explicit proof. Let $\beta : B \to \mathbb{P}^1$ and τ_B be an involution of B acting non-trivially on $\mathbb{P}^1$. Then, there exists an involution $\alpha_B : B \to B$ which maps fibers of B to fibers with the property that

$$\alpha_B(e) = (e)\,, \tag{3.11}$$

where e is the zero section of B, and a section $\xi \in \Gamma(B)$ satisfying

$$\alpha_B(\xi) = (-1)_B(\xi)\,, \tag{3.12}$$

such that

$$\tau_B = t_\xi \circ \alpha_B\,. \tag{3.13}$$

First note that τ_B is indeed an involution. To see this, use condition (3.12) to show that

$$t_\xi \circ \alpha_B = \alpha_B \circ t_{-\xi}\,. \tag{3.14}$$

It follows immediately that $\tau_B^2 = id$ and, hence, τ_B is an involution on B.

What are the properties of α_B and ξ ? Note that (3.2) and (3.13) imply

$$\beta \circ \alpha_B = \tau_{\mathbb{P}^1} \circ \beta \tag{3.15}$$

and, therefore, the action of α_B on $\mathbb{P}^1$ is determined by $\tau_{\mathbb{P}^1}$. As mentioned above, each non-trivial involution $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$ has two fixed points, which we call 0 and ∞. Therefore, any involution α_B and, hence, any involution of the form (3.13) leave two fibers stable, the fiber $f_0 = \beta^{-1}(0)$ and the fiber $f_\infty = \beta^{-1}(\infty)$. Next consider a section $\xi \in \Gamma(B)$ satisfying (3.12). This equation, indeed, puts some restrictions on ξ, but to compute these it is necessary to have more information about the action of α_B. We will return to this important point at the end of this section.

To understand the involutions with non-trivial $\tau_{\mathbb{P}^1}$ better, we will now analyze how α_B acts on the Weierstrass representation W_B of B in P. Recall from (2.35) that W_B is given by the equation

$$y^2 z = x^3 + g_2 x z^2 + g_3 z^3\,, \tag{3.16}$$

where, using the identity $H^0(P, p^*\mathcal{L}^n) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{L}^n)$ for any integer n, (2.34) becomes

$$g_2 \in H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)\right), \qquad g_3 \in H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)\right). \tag{3.17}$$

Furthermore, it is easy to see that the zero section e is given by the solution of (3.16) for

$$x = z = 0\,. \tag{3.18}$$

It is straightforward to find the action of the natural involution $(-1)_B$ on W_B. It is

$$(-1)_{W_B} : (x : y : z) \to (x : -y : z)\,. \tag{3.19}$$

Clearly this action is an involution, which leaves each fiber stable and the zero section invariant. That is, it represents $(-1)_B$ on W_B. Recall that on a generic surface B, there are no sections whose intersection point with every fiber is a non-trivial point of order two. However, there exists a tri-section, that is, a divisor on B which intersects each fiber in the three points e', e'' and e'''. It can be found by solving (3.16) with

$$y = 0\,. \tag{3.20}$$

Let us now find the action of α_B on W_B. First consider the bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. The non-trivial involution $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ allows one to construct a second bundle, the pull-back $\tau^*_{\mathbb{P}^1}\mathcal{O}_{\mathbb{P}^1}(1)$, along with a natural bundle map

$$t_{\mathbb{P}^1} : \tau^*_{\mathbb{P}^1}\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)\,. \tag{3.21}$$

Note that, in general, $\tau^*_{\mathbb{P}^1}\mathcal{O}_{\mathbb{P}^1}(1)$ need not be isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. However, recall that the projective coordinates t_0 and t_1 on $\mathbb{P}^1$ form a basis of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Therefore, the action (3.4) of $\tau_{\mathbb{P}^1}$ on t_0, t_1 lifts to an involution on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. One can show that this induces a bundle isomorphism

$$\gamma : \tau^*_{\mathbb{P}^1}\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)\,, \tag{3.22}$$

where γ acts linearly on the fibers and trivially on $\mathbb{P}^1$. Combining this result with (3.21), it follows that the map

$$\tau_{\mathbb{P}^1} : \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)\,, \tag{3.23}$$

where $\tau_{\mathbb{P}^1} = t_{\mathbb{P}^1} \circ \gamma^{-1}$, is an involution on $\mathcal{O}_{\mathbb{P}^1}(1)$. That is, $\tau_{\mathbb{P}^1}$ is the lift to $\mathcal{O}_{\mathbb{P}^1}(1)$ of the involution $\tau_{\mathbb{P}^1}$ on $\mathbb{P}^1$. This can be extended to any bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ with non-negative integer k using the fact that

$$H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)\right) = S^k H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\right)\,, \tag{3.24}$$

where S^k denotes the k-th symmetric product. As above, this implies an isomorphism between $\tau^*_{\mathbb{P}^1}\mathcal{O}_{\mathbb{P}^1}(k)$ and $\mathcal{O}_{\mathbb{P}^1}(k)$. In combination with the natural bundle map, this induces the involution

$$\tau_{\mathbb{P}^1} : \mathcal{O}_{\mathbb{P}^1}(k) \to \mathcal{O}_{\mathbb{P}^1}(k)\,, \tag{3.25}$$

which is the lift of $\tau_{\mathbb{P}^1}$ to $\mathcal{O}_{\mathbb{P}^1}(k)$. Recalling from (2.27) and (2.28) that

$$P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))\,, \tag{3.26}$$

we see using (3.25) that $\tau_{\mathbb{P}^1}$ induces an action on P, which we denote by τ_P. That is,

$$\tau_P : P \to P\,. \tag{3.27}$$

This involution lifts again to an involution

$$\tau_P : \mathcal{O}_P(r) \to \mathcal{O}_P(r)\,, \tag{3.28}$$

where r is a non-negative integer. The bundle $\mathcal{O}_P(1)$ was discussed in the previous section. Finally, note that the actions (3.25) and (3.28) induce a natural involution

$$\tau_P : H^0(P, \mathcal{O}_P(r) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(s)) \to H^0(P, \mathcal{O}_P(r) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(s)) , \tag{3.29}$$

for any non-negative integers r and k.

We can now give the action of α_B on B in terms of the Weierstrass model W_B and τ_P. We refer the reader to [10] for a complete discussion. Let us assume that B admits an involution α_B. Then, there must exist a unique involution

$$\alpha_{W_B} : W_B \to W_B \tag{3.30}$$

satisfying

$$\alpha_{W_B} \circ \nu = \nu \circ \alpha_B , \tag{3.31}$$

where $\nu : B \to W_B$ is the natural map of B into P. It was shown in [10] that α_{W_B} must be either of the form

$$\alpha_{W_B} = \tau_P|_{W_B} \tag{3.32}$$

or

$$\alpha_{W_B} = \tau_P|_{W_B} \circ (-1)_{W_B} . \tag{3.33}$$

An important implication of this result is that, for α_{W_B} and, hence, α_B to exist, the Weierstrass model $W_B \subset P$ must be stable under involution τ_P, that is

$$\tau_P|_{W_B} : W_B \to W_B . \tag{3.34}$$

This requirement puts significant restrictions on the allowed surfaces B, as we now show.

Recall from (2.29) that $z \in H^0(\mathcal{O}_P(1))$. Since

$$H^0(P, \mathcal{O}_P(1)) = H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)\right) , \tag{3.35}$$

we see that the section z must be the constant section of $\mathcal{O}_{\mathbb{P}^1}$. Now, it follows from (3.29) with $r = 1, s = 0$ that $\tau_P : H^0(P, \mathcal{O}_P(1)) \to H^0(P, \mathcal{O}_P(1))$. Since z is a constant section over $\mathbb{P}^1$, it must be invariant under τ_P. That is

$$\tau_P z = z . \tag{3.36}$$

In a similar way, one can show that

$$\tau_P x = x , \qquad \tau_P y = y . \tag{3.37}$$

Since W_B is stable under τ_P and x, y, z are invariant, it follows from (3.16) that g_2 and g_3 must satisfy

$$\tau_{\mathbb{P}^1} g_2 = g_2 , \qquad \tau_{\mathbb{P}^1} g_3 = g_3 . \tag{3.38}$$

Therefore, the existence of α_B on B implies that the sections g_2, g_3 of the corresponding Weierstrass model must satisfy (3.38). The converse is also true, that is, if (3.38) is satisfied then there exists an associated involution α_B on B. Note that (3.38) is a strong restriction on the Weierstrass model W_B and, hence, on the structure of B.

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Expression (3.32) and (3.33) allow us to establish a very important property of involutions α_B on B, namely that α_B must fix the fiber $f_0 = \beta^{-1}(0)$ point-wise and have four isolated fixed points on the fiber $f_\infty = \beta^{-1}(\infty)$. These four fixed points are e, e', e'' and e'''. To prove this, consider the Weierstrass representation W_B and assume that $\alpha_{W_B} = \tau_P|_{W_B}$. Let us first focus on f_0.

Recall that $p^{-1}(0)$ is the fiber in P over 0. It follows from (3.26) that

$$P|_{p^{-1}(0)} = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}|_0 \oplus \mathcal{O}_{\mathbb{P}^1}(2)|_0 \oplus \mathcal{O}_{\mathbb{P}^1}(3)|_0\right). \tag{3.39}$$

Note that $P|_{p^{-1}(0)}$ is isomorphic to $\mathbb{P}^2$. Since $t_1 = 0$ at $0 = (1:0)$, we see that

$$1\,, \quad t_0(0)^2\,, \quad t_0(0)^3 \tag{3.40}$$

are the projective coordinates on $P|_{p^{-1}(0)}$. However, it follows from (3.4) that $\tau_{\mathbb{P}^1} t_0(0) = t_0(0)$ and, hence, that

$$\tau_P P|_{p^{-1}(0)} = P|_{p^{-1}(0)} \tag{3.41}$$

point-wise. Therefore, $\alpha_{W_B} = \tau_P|_{W_B}$ leaves the fiber $f_0 = W_B \cap P|_{p^{-1}(0)}$ fixed point-wise, as claimed.

Now consider f_∞. Again, f_∞ is described by the vanishing locus of a homogeneous cubic polynomial, this time in $P|_{p^{-1}(\infty)}$ where

$$P|_{p^{-1}(\infty)} = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}|_\infty \oplus \mathcal{O}_{\mathbb{P}^1}(2)|_\infty \oplus \mathcal{O}_{\mathbb{P}^1}(3)|_\infty\right). \tag{3.42}$$

Since $t_0 = 0$ at $\infty = (0:1)$, we see that

$$1, \;\; t_1(\infty)^2\,, \quad t_1(\infty)^3 \tag{3.43}$$

are the projective coordinates on $P|_{p^{-1}(\infty)}$. From (3.4), we know that $\tau_{\mathbb{P}^1} t_1(\infty) = -t_1(\infty)$ and, hence, that the invariant points of $P|_{p^{-1}(\infty)}$ must satisfy

$$t_1(\infty)^3 = 0\,. \tag{3.44}$$

Recalling from (3.13) that z is a section of $\mathcal{O}_{\mathbb{P}^1}$ and, similarly, that x and y are sections of $\mathcal{O}_{\mathbb{P}^1}(2)$ and $\mathcal{O}_{\mathbb{P}^1}(3)$ respectively, we see that (3.44) is fulfilled when

$$y|_{p^{-1}(\infty)} = 0\,. \tag{3.45}$$

This, however, when inserted into (3.16) leads precisely to the equation for the four fixed points e, e', e'', e'''. We conclude, therefore, that $\alpha_{W_B} = \tau_P|_{W_B}$ leaves the fiber $f_\infty = W_B \cap P|_{p^{-1}(\infty)}$ fixed on the four isolated points e, e', e'' and e''', as claimed. Note that if the involution α_{W_B} is of the form (3.33), we get the same distribution of fixed points with f_0 and f_∞ switched. That is, we get four isolated fixed points on f_0 and a trivial action on f_∞. Therefore, after a possible relabeling of f_0 and f_∞, we can assume that α_{W_B} is of the form (3.32), which we do henceforth.

Having established these results, we can now return to the question of the constraints on sections $\xi \in \Gamma(B)$ satisfying equation (3.12). It follows from the fact that α_B leaves fiber f_0 point-wise fixed that

$$\alpha_B|_{f_0} = \mathrm{id}\,. \tag{3.46}$$

Hence, restricted to f_0, equation (3.12) becomes

$$\xi(0) = -\xi(0)\,. \tag{3.47}$$

Therefore, at f_0, any such section must pass through either the zero point e or one of the remaining three invariant points of $(-1)_B$, namely, $e^{'}, e^{''}$ and $e^{'''}$. Now note that

$$\alpha_B \circ (-1)_B = (-1)_B \circ \alpha_B\,. \tag{3.48}$$

Therefore, if ξ is a solution of (3.12), then so is $(-1)_B(\xi)$. We conclude that the solutions of equation (3.12) come in pairs

$$(\xi, (-1)_B(\xi))\,, \tag{3.49}$$

where

$$\xi(0) = e, e^{'}, e^{''} \text{or } e^{'''}\,. \tag{3.50}$$

The only obvious solution is $\xi = e$, which corresponds to $\tau_B = \alpha_B$. However, it is not clear that any other such sections ξ exist. We will show later in the paper that any surface B that admits an involution α_B, also admits sections $\xi \neq e$ which satisfy (3.12).

In principle, all three types of involutions on B, of the form (3.8), (3.10) and (3.13) respectively, are of interest. However, in this paper we are interested in Calabi-Yau threefolds X elliptically fibered over a base $B^{'} = d\mathbb{P}_9$. It will be shown in section 7 that such threefolds are always a fiber product over $\mathbb{P}^1$ of the base $B^{'}$ with second rational elliptic surface B. That is, $X = B \times_{\mathbb{P}^1} B^{'}$. Suffice it here to say that any involution on X can always be constructed as the lift of involutions on B and $B^{'}$. Furthermore, any involution on X involving the lift of an involution on B, $B^{'}$ of the form (3.8) must necessarily have at least one fixed point. This will be discussed in [17]. Since, in this paper, we are interested in freely acting involutions on X we, henceforth, do not consider involutions of type (3.8). Happily, involutions on X constructed solely from the remaining two types (3.10) or (3.13) where $\xi \neq e$ can have fixed point free actions. Recall, however, that involutions of the form (3.10) require a section which intersects each fiber at a point of order two. Such sections, as we will show, only exists in special circumstances and, hence, are of secondary importance. We turn, therefore, to the remaining involutions of the form (3.13).

4. The quotient B/α_B

In this section, we analyze rational elliptic surfaces B admitting an involution α_B. Recall from section 3, that the existence of α_B restricts the form of their Weierstrass model W_B. Note that such models are only smooth for generic surfaces. Since, as shown later, we need to consider surfaces with singular Weierstrass models, and since we must discuss the induced action of α_B on the cohomology $H_2(B, \mathbb{Z})$, we present, in this section, yet another description of such surfaces B.

Let us assume the surface B admits an involution α_B. We now show that, in order to understand such surfaces B, it is sufficient to study the quotient B/α_B. First of all, since α_B maps fibers of B to fibers, it is clear that B/α_B is a torus fibration over the base

$\mathbb{P}^1/\tau_{\mathbb{P}^1} \cong \mathbb{P}^1$. We denote the projection map by $\beta_{\alpha_B} : B/\alpha_B \to \mathbb{P}^1$ and get the following commuting diagram

$$\begin{tikzcd} B \arrow[r, "\kappa"] \arrow[d, "\beta"'] & B/\alpha_B \arrow[d, "\beta_{\alpha_B}"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{sq}"'] & \mathbb{P}^1 \end{tikzcd},$$

where κ and sq are the double covering maps associated with modding out α_B and $\tau_{\mathbb{P}^1}$ respectively. Furthermore, since α_B preserves e, the curve e/α_B is a zero section of B/α_B. Hence, B/α_B is an elliptic fibration. Note, that we will call the image of 0 and ∞ under sq in $\mathbb{P}^1/\tau_{\mathbb{P}^1}$ 0 and ∞ as well.

Secondly, since α_B is an involution of B it must map singular fibers to singular fibers. Recall from section 2 that, generically, B possesses twelve I_1 fibers. Therefore, α_B must map pairs of I_1 fibers into each other. It follows that B/α_B has six I_1 fibers, each the result of identifying a pair of I_1 fibers on B.

Are there other degenerate fibers? To answer this question, we must explore the singularity structure of B/α_B. In the previous section, we showed that α_B leaves the fiber $f_0 = \beta^{-1}(0)$ point-wise fixed. Furthermore, α_B leaves four isolated points e, e', e'', e''' fixed on $f_\infty = \beta^{-1}(\infty)$. Since the fixed point locus of α_B in f_0 is of codimension one in B, the associated fiber f_0/α_B in B/α_B is a smooth elliptic fiber. For simplicity denote f_0/α_B by f_0. Note that f_0 in B/α_B is given by $\beta_{\alpha_B}^{-1}(0)$. However, f_∞/α_B, with its four fixed points each of codimension two in B, is more interesting. It is easy to see that the local neighborhood in B/α_B of any one such fixed point, call it p, is a complex surface in $\mathbb{C}^3$ with coordinates u, v and w where $p = (0,0,0)$. The precise embedding is described by the equation

$$w^2 = uv\,. \tag{4.1}$$

It is well known that this relation describes a surface with an isolated singular point of type A_1 at $p = (0,0,0)$. It follows that f_∞/α_B in B/α_B is a degenerate fiber. Since α_B leaves f_∞ stable, that is $\alpha_B : f_\infty \to f_\infty$, and has four fixed points, the map $f_\infty \to f_\infty/\alpha_B$ is a double cover branched over four points. As we show later in this section, this implies that $f_\infty/\alpha_B \cong \mathbb{P}^1$ is a line. Actually, as a fiber in B/α_B it is a double line passing through the four isolated A_1 singularities. This is illustrated in figure 2. Henceforth, for simplicity, we will denote f_∞/α_B by f_∞. Note that f_∞ in B/α_B is given by $\beta_{\alpha_B}^{-1}(\infty)$. Hence, B/α_B is not a smooth surface. Rather, it has four isolated A_1 singularities on its f_∞ fiber.

Although B/α_B is the surface of primary interest, it is useful to consider the related manifold $\widehat{B/\alpha_B}$, obtained from B/α_B by blowing-up each of these four A_1 singularities into a $\mathbb{P}^1$. The result is a smooth elliptic fibration which has six I_1 fibers. However, one can show that the double fiber with four isolated A_1 singularities will become a I_0^* Kodaira fiber once the singularities are resolved. This is indicated in figure 2. Thus, $\widehat{B/\alpha_B}$ is a smooth elliptic fibration with seven degenerate fibers, six I_1 and one I_0^*. Recall from (2.16) that $\chi(I_1) = 1$. Furthermore, it is well-known that

$$\chi(I_0^*) = 6\,. \tag{4.2}$$

It follows that the Euler characteristic of $\widehat{B/\alpha_B}$ is

$$\chi(\widehat{B/\alpha_B}) = 6\chi(I_1) + \chi(I_0^*) = 12 \tag{4.3}$$

and, hence, $\widehat{B/\alpha_B}$ is a rational elliptic surface. That is, $\widehat{B/\alpha_B} = d\mathbb{P}_9$. The singular fibration $\beta_{\alpha_B} : B/\alpha_B \to \mathbb{P}^1$ is its Weierstrass model.

We can summarize the relationship of B/α_B to B as follows. The map

$$\kappa : B \to B/\alpha_B \tag{4.4}$$

is a double cover of B/α_B. However, since f_0 is point-wise fixed and e, e', e'', e''' on f_∞ are fixed under α_B, it follows that κ has a branch locus in B/α_B consisting of all of f_0 and the four singular points in f_∞. That is, the double cover of B/α_B is branched at all these points. It was shown in [10] that for a given rational elliptic surface $\widehat{B/\alpha_B}$ with six I_1 fibers and one I_0^* fiber, B is the unique double cover of B/α_B branched over the fiber f_0 and the four singular points in f_∞. Therefore, a given B/α_B determines B and α_B.

Thus far, we have discussed generic properties of the quotient B/α_B. To go further, we will give a complete geometric construction of B/α_B. Specifically, every surface B/α_B can be obtained as a double cover of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ branched along a curve $M \subset Q$ of bidegree $(2,4)$ which splits as a union of two curves T and r, of bidegree $(1,4)$ and $(1,0)$ respectively. That is

$$M = T \cup r\,. \tag{4.5}$$

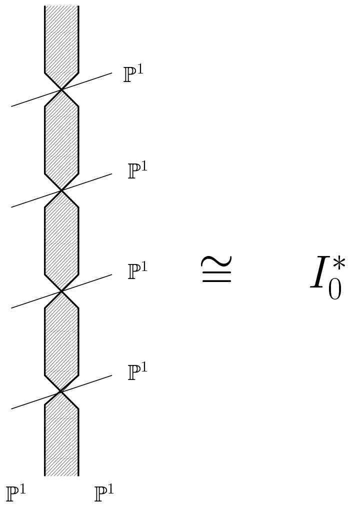

Figure 2: A double $\mathbb{P}^1$ line with four surface A_1 singularities. Its resolution is an I_0^* Kodaira fiber. The cross-hatch indicates the identification of the two $\mathbb{P}^1$ lines.

To see this, first define the projection maps

$$p_i : Q \to \mathbb{P}^1_i \tag{4.6}$$

for $i = 1, 2$ where $\mathbb{P}^1_i$ is the i-th $\mathbb{P}^1$ factor in $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Let z_i be the affine coordinates of $\mathbb{P}^1_i$ for $i = 1, 2$ respectively. In terms of these coordinates, the curve $r \subset Q$ of bidegree $(1,0)$ must satisfy the linear equation

$$-a + z_1 = 0\,, \tag{4.7}$$

where a is a complex number. Hence, $z_1 = a$. We refer the reader to figure 3 for a pictorial representation of the curve r. The curve $T \subset Q$ of bidegree $(1,4)$, on the other hand, must satisfy an equation that is linear in z_1 and of degree four in z_2. A generic curve of this type is pictured in figure 3 as well.

Clearly, for the curve r located at fixed $z_1 = a$, T will generically intersect r at four distinct points. In this paper, we will assume that T is smooth and intersects r transversely. Having specified Q and the curve M, let us now construct the double cover. Denote by

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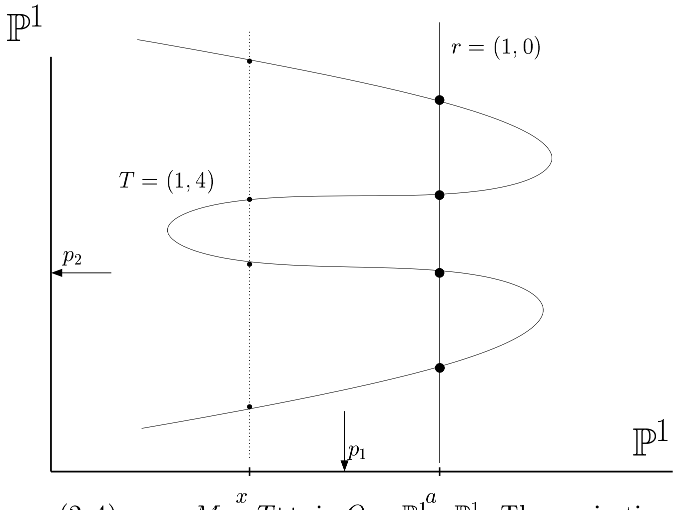

Figure 3: The bidegree $(2,4)$ curve $M = T \cup r$ in $Q = \mathbb{P}^1 \times \mathbb{P}^1$. The projections $p_i : Q \to \mathbb{P}^1_i, i = 1,2$ are explicitly shown.

W_M the double cover of Q branched along the curve M, and denote the double cover map by π. That is

$$\pi : W_M \to Q\,. \tag{4.8}$$

Combining this with (4.6), we see that there are two natural projection maps

$$\tilde{p}_i : W_M \to \mathbb{P}^1_i \tag{4.9}$$

for $i = 1,2$, where

$$\tilde{p}_i = p_i \circ \pi\,. \tag{4.10}$$

We will be primarily interested in $\tilde{p}_1 : W_M \to \mathbb{P}^1_1$. We begin our analysis of W_M by considering a generic point $x \in \mathbb{P}^1_1$, where $x \neq a$. Then $p_1^{-1}(x)$ is a $\mathbb{P}^1$, pictorially given by the dashed line in figure 3, that intersects the curve T at four points. Its pre-image in W_M, $\tilde{p}_1^{-1}(x)$, is then the double cover of $p_1^{-1}(x) \cong \mathbb{P}^1$ branched over the four points where $p_1^{-1}(x)$ intersects T. We now show this is a torus T^2.

Having a double cover $\mathcal{C}$ of $\mathbb{P}^1$ which is branched over 4 points means that there is a map

$$c : \mathcal{C} \to \mathbb{P}^1 \tag{4.11}$$

such that the generic fiber of c contains two points. However, there are four special points $b_i, i = 1, \ldots, 4$ in $\mathbb{P}^1$ whose pre-images consist of only a single point R_i. The points b_i are called branch points and the points R_i in $\mathcal{C}$ are called ramification points. Here, since only two points get identified to a single point, they are called simple ramification points. Now $\mathcal{C}$ can be described as follows. Take two copies of $\mathbb{P}^1$ lines, we will call them the upper and lower sheets, and fix four points $p_1, \ldots, p_4$ on each of them. Order the points pairwise and cut the line between the points of each pair, as shown in figure $4a$. Open the cuts and identify the upper and lower sheets along them. The result of doing this is a torus T^2. This is pictured in figure $4b$. It follows that

$$\tilde{p}_1 : W_M \to \mathbb{P}^1_1 \tag{4.12}$$

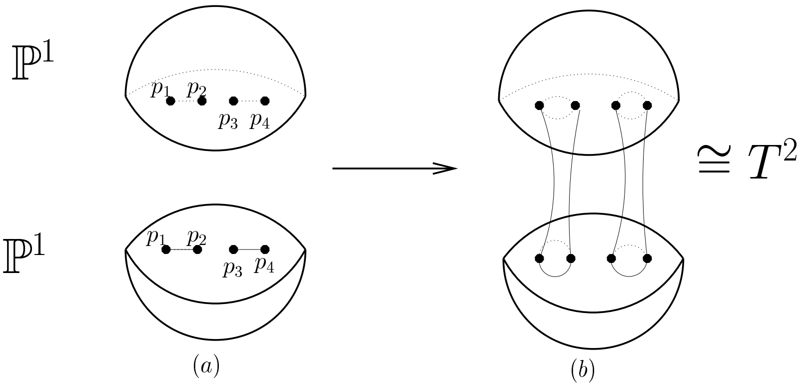

Figure 4: The double cover of $\mathbb{P}^1$ branched over four points.

is a torus fibration, since the fiber $\tilde{p}_1^{-1}(x)$ over a generic point $x \in \mathbb{P}^1$ is a torus.

Are there any bad fibers in W_M ? To explore this, we have to analyze more closely the structure of the curve $T \subset Q$. A naive form of T was illustrated in figure 3. Consider, for a moment, the projection $p_2 : Q \to \mathbb{P}^1_2$. Clearly, p_2 restricted to T is a degree one cover of $\mathbb{P}^1$ and, hence,

$$\chi(T) = \chi(\mathbb{P}^1) = 2\,. \tag{4.13}$$

Now consider $p_1 : Q \to \mathbb{P}^1_1$. Under p_1, T is clearly a degree four cover of $\mathbb{P}^1$. Then the Riemann-Hurwitz formula states that for simple ramification points of T we have

$$\chi(T) = 4\chi(\mathbb{P}^1 \setminus \{b_i\}) + 3A\,, \tag{4.14}$$

where the 4 arises since T is a degree four cover and $\{b_i\}$ for $i = 1, \ldots, A$ are the branch points of the simple ramification points of T over $\mathbb{P}^1_1$. For more information about the Riemann-Hurwitz formula, we refer the reader to [19]. By definition, above each such point T intersects $p_1^{-1}(b_i)$ in only three, not four, points. At one of these points, the ramification point R_i, the fiber $p_1^{-1}(b_i)$ intersects T tangentially. This is shown in figure 5. Therefore, the branch locus for the double cover $\tilde{p}_1^{-1}(b_i)$ consists of three instead of four points. Looking back to figure 4, we see that over the points b_i one of the pairs, for example (p_1, p_2) degenerates to a single point. This implies that one of the A-cycles of the torus shrinks to zero. Therefore, as discussed earlier, one obtains a I_1 fiber. Since the curve M is smooth at this ramification point, the singularity is in the fiber only and not in W_M. Therefore, we need not blow-up this point. Combining (4.13) and (4.14), we can solve for the number A of such branch points. We find that

$$A = 6\,. \tag{4.15}$$

We conclude that there are six I_1 fibers in W_M that arise in this manner.

Are there other degenerate fibers? Let us consider the specific point $a \in \mathbb{P}^1_1$. Then, $p_1^{-1}(a) = r$ is a $\mathbb{P}^1$ marked at the four points where it intersects T. Hence, one might imagine that $\tilde{p}_1^{-1}(a)$ is simply two copies of $\mathbb{P}^1$ identified at the four marked points and, hence, is a smooth torus. However, one must keep in mind that $p_1^{-1}(a) = r$ is itself in

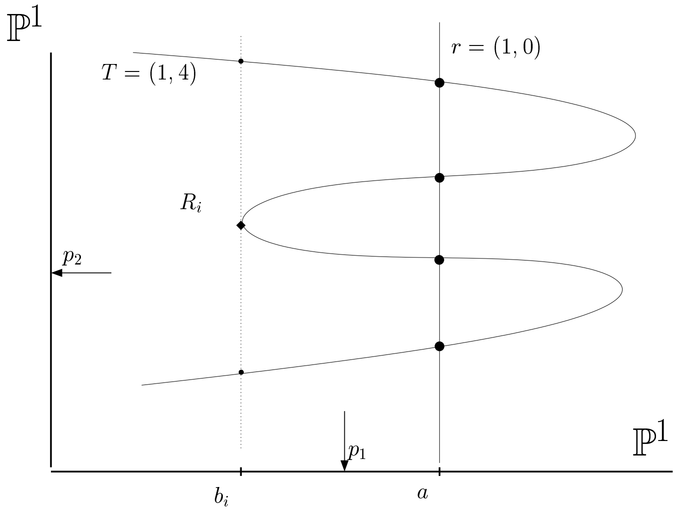

Figure 5: A schematic representation of the curve T on $Q = \mathbb{P}^1 \times \mathbb{P}^1$ indicating a simple ramification point R_i over the branch point b_i in $\mathbb{P}^1$.

the branch locus and, hence, the two $\mathbb{P}^1$ lines must be identified as well. This produces a double line with four isolated singular points of type A_1. Unlike the I_1 fiber singularity at a ramification point, each of these four points is singular in W_M. This follows from the assumption that the curves T and r intersect, so the branch locus for W_M is not smooth at these points. More precisely, since these curves intersect transversally at these points, the singularity is an A_1 surface singularity. This has precisely the properties of the f_∞ fiber in B/α_B discussed earlier that can be resolved into I_0^*.

To finish our analysis, let us chose any one of the four points of intersection of curves T and r, and let u be a curve of bidegree $(0,1)$ passing through that point. Then $\pi^{-1}(u)$ is two copies of $\mathbb{P}^1$ intersecting at that point. Clearly, each one of these $\mathbb{P}^1$ lines is a section of the fibration (4.12). Choose either copy as the zero section. Hence, the fibration (4.12) is an elliptic fibration. Putting everything together, we see that W_M in (4.12) is an elliptic fibration with six I_1 fibers and one fiber consisting of a double line with four isolated A_1 type surface singularities which can be blown-up to an I_0^* Kodaira fiber. We conclude that

$$W_M \cong B/\alpha_B \,, \tag{4.16}$$

as claimed above. Note, that this allows us to identify the point 0 and the fiber f_0 in W_M and, hence, in Q.

This characterization of B/α_B as the double cover of Q branched over the curve M allows us to analyze the properties of B/α_B in detail. In particular, we can compute the dimension of the moduli space of B/α_B. It is clear from the above construction that, up to some caveats we discuss below, the moduli of W_M is the same as the moduli of the curve $M = T \cup r$ in Q. Consider some curve C of bidegree (n_1, n_2). Then, the number of parameters for C is given by the dimension of the space

$$|\mathcal{O}_Q(n_1,n_2)| \equiv \mathbb{P}H^0(Q, \mathcal{O}_Q(n_1,n_2)) \,, \tag{4.17}$$

that is, the projectivization of the space of global sections of $\mathcal{O}_Q(n_1, n_2)$. These sections are specified by a polynomial of degree n_1 in the coordinate z_1 of $\mathbb{P}^1_1$ and of degree n_2 in the coordinate z_2 of $\mathbb{P}^1_2$, which have $n_1 + 1$ and $n_2 + 1$ parameters respectively. It follows that

$$\dim |\mathcal{O}_Q(n_1, n_2)| = (n_1 + 1)(n_2 + 1) - 1 . \tag{4.18}$$

Note that one must subtract 1 in (4.18) since one parameter can always be scaled away. We can now consider the component curves of M. The curve r has bidegree $(1, 0)$. Hence, using (4.18), we see that

$$\dim |\mathcal{O}_Q(1, 0)| = 1 . \tag{4.19}$$

This is consistent with equation (4.7) and simply tells us that the complex constant a is the modulus of r. The curve T has bidegree $(1, 4)$. It follows from (4.18) that

$$\dim |\mathcal{O}_Q(1, 4)| = 9 . \tag{4.20}$$

One might imagine that the total number of moduli of curve $M = T \cup r$ is simply the sum of (4.19) and (4.20). However, one must be careful to subtract out the automorphisms of $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Since $\mathrm{Aut}(\mathbb{P}^1) = \mathbb{P}GL(2)$, it follows that $\dim\mathrm{Aut}(\mathbb{P}^1) = 3$ and hence

$$\dim \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) = 6 . \tag{4.21}$$

Furthermore, recall that in addition to f_∞, which is specified by the location modulus a of the curve r, there is the invariant fiber f_0 in B/α_B. The location of this zero point in $\mathbb{P}^1$ is yet another complex modulus. We conclude, therefore, that the moduli of $W_M \cong B/\alpha_B$ are the moduli of M reduced by the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ but enhanced by the location modulus of fiber f_0. That is

$$\dim \mathcal{M}\,(B/\alpha_B) = 1 + 9 - 6 + 1 = 5 , \tag{4.22}$$

where $\mathcal{M}(B/\alpha_B)$ stands for the moduli space of B/α_B. Recall from (4.4) that B is the double cover of B/α_B branched over f_0 and the four singular points in f_∞. Since this double cover is completely determined by the above quantities, it has the same moduli space. Therefore, we have shown that

$$\dim \mathcal{M}(B) = 5 , \tag{4.23}$$

where $\mathcal{M}(B)$ denotes the moduli space of B.

In this section, we have given a rather complete characterization of the rational elliptic surfaces B that admit an involution α_B which leaves the zero section invariant and acts non-trivially on the base $\mathbb{P}^1$. The space of such surfaces is specified by five parameters. Recalling from (2.8) that generic rational elliptic surfaces are specified by eight parameters, we see that the existence of such involution puts severe restrictions on B. From now on we will restrict ourselves to rational elliptic surfaces which are contained in this 5 dimensional family and, hence, an involution α_B will always exist.

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5. Surfaces with a $\mathbb{Z}_2$ group of automorphisms

In the previous section, we discussed rational elliptic surfaces with an involution α_B leaving the zero section fixed. But, as stated at the end of section 3, to construct freely acting involutions on the Calabi-Yau threefolds X we need to find sections $\xi \neq e$ which fulfill condition (3.12). It was shown in [10] that for each member of the five dimensional family of rational elliptic surfaces just discussed, there is a rank four lattice of such sections. Since these are the surfaces which have a least one involution which can lead to a freely acting involution on the Calabi-Yau space X, we will call them surfaces with a $\mathbb{Z}_2$ group of automorphisms.

We will further restrict ourselves to a four dimensional sub-family of surfaces with a $\mathbb{Z}_2$ group of automorphisms. We do so for the following two reasons. First, the restriction to this sub-family will introduce new fiber classes. As shown in [11], these classes are a necessary ingredient to construct stable, rank five vector bundles on X which lead to standard models in string theory compactifications. Secondly, for the surfaces B of this four dimensional family, we will be able to solve explicitly for some sections ξ.

Consider the five dimensional family of surfaces B discussed so far. This family was obtained by constructing $B/\alpha_B \cong W_M$ as the double cover of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve $M \subset Q$ of bidegree $(2,4)$ which split as $M = T \cup r$, where T is a smooth curve of bidegree $(1,4)$, r has bidegree $(1,0)$ and we have assumed that T and r intersect transversely. In this section, we will further restrict B by adding the requirement that

$$T = t \cup s \,, \tag{5.1}$$

where the curve t has bidegree $(1,3)$ and the curve s has bidegree $(0,1)$. That is, we henceforth assume

$$M = t \cup s \cup r \,. \tag{5.2}$$

In terms of the affine coordinates z_i in $\mathbb{P}^1_i$ for $i = 1,2$, the curve $s \subset Q$ of bidegree $(0,1)$ must satisfy the linear equation

$$-b + z_2 = 0 \,, \tag{5.3}$$

where b is a complex constant. Hence, $z_2 = b$. We refer the reader to figure 6 for a pictorial representation of the curve s, as well as the curve r of bidegree $(1,0)$ given by (4.7). The curve t of bidegree $(1,3)$, on the other hand, must satisfy an equation that is linear in z_1 and is of degree three in z_2. A generic such curve is described in figure 6. Note that the curves t and s intersect at a single point, which we will denote by p. Clearly, for the curve r located at fixed $z_1 = a$, s will intersect r at one point whereas the curve t will generically intersect r at three points. As before, we will assume that these intersections are transversal. Having specified the curve M in (5.2), we now analyze the double cover W_M of Q branched over M. As discussed above, W_M is a fibration

$$\tilde{p}_1 : W_M \to \mathbb{P}^1_1 \tag{5.4}$$

over $\mathbb{P}^1$.

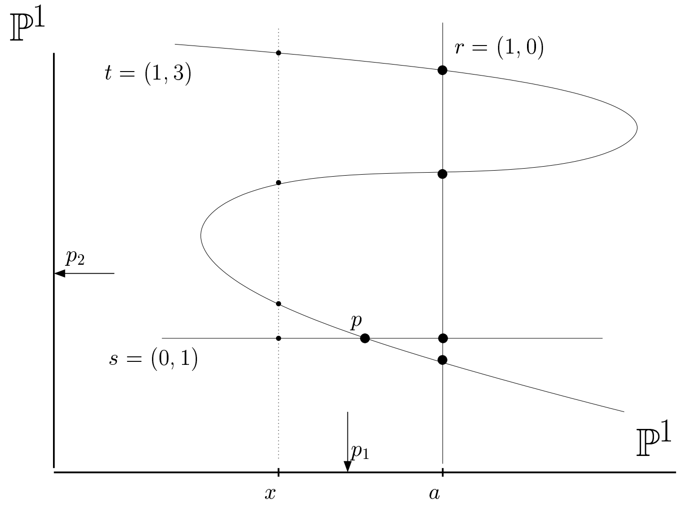

Figure 6: The bidegree (2,4) curve $M = t \cup s \cup r$ in $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

We begin by considering a generic point $x \in \mathbb{P}^1_1$ where $x \neq a$. Then $p_1^{-1}(x)$ is a $\mathbb{P}^1$ that intersects the curve $T = t \cup s$ at four marked points. Exactly as in the previous section, $\tilde{p}_1^{-1}(x)$ is a double cover of $\mathbb{P}^1$ branched at four points and, hence, is a torus. It follows that (5.4) is a torus fibration, since the fiber $\tilde{p}_1^{-1}(x)$ over a generic point $x \in \mathbb{P}^1_1$ is a torus.

To explore possible degenerate fibers in W_M, we have to analyze more closely the structure of the curves t and s in Q. Consider, for a moment, the projection $p_2 : Q \to \mathbb{P}^1_2$. Clearly, $p_2 : t \to \mathbb{P}^1$ is a degree one cover and, hence, $t \cong \mathbb{P}^1$. Then

$$\chi(t) = \chi(\mathbb{P}^1) = 2\,. \tag{5.5}$$

Now consider $p_1 : Q \to \mathbb{P}^1_1$. Under p_1, t is clearly a three-fold cover of $\mathbb{P}^1$ with generically simple ramification points. Then the Riemann-Hurwitz formula states that

$$\chi(t) = 3\chi\left(\mathbb{P}^1 \setminus \{b_i\}\right) + 2A\,, \tag{5.6}$$

where the 3 arises since t is a three-fold cover and $\{b_i\}$ for $i = 1, \ldots, A$ is the set of branch points of t in $\mathbb{P}^1_1$ corresponding to the simple ramification points. By definition, each such branch point is an isolated point in $\mathbb{P}^1$ above which t intersects $p_1^{-1}(b_i)$ in only two, not three, points. Combining (5.5) and (5.6), we can solve for the number A of such ramification points. We find that

$$A = 4\,. \tag{5.7}$$

Let b_i be such a point in $\mathbb{P}^1$. As discussed previously, the fiber $\tilde{p}_1^{-1}(b_i)$ over such a point is branched over three instead of four points and, hence, $\tilde{p}_1^{-1}(b_i)$ is singular and of Kodaira type I_1. Since curve M is smooth, this is not a singularity in W_M and need not be blown-up. We conclude that there are four I_1 fibers that arise in this manner.

Are there other degenerate fibers? Let us consider the intersection point p of the curves t and s. This projects to a point $p_1(p)$ in $\mathbb{P}^1$ and lies in the fiber $p_1^{-1}(p_1(p))$ in Q. Note from figure 7 that $p_1^{-1}(p_1(p))$ intersects t alone at two points and both t and s

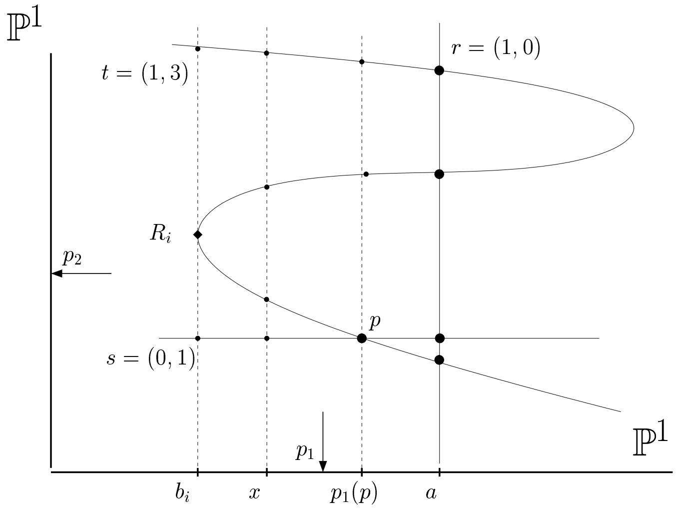

Figure 7: A schematic representation of curve $M = t \cup s \cup r$ showing a simple ramification point R_i explicitly. We indicate the fibers over a generic point x, a branch point b_i, $p_1(p)$ and a.

simultaneously at the point p. It follows that $\tilde{p}_1^{-1}(p_1(p))$ is a double cover of $\mathbb{P}^1$ in W_M branched at three points. Furthermore, since both t and s are in the branch locus, the branch locus of W_M has an ordinary nodal singularity at p. Therefore, at this point W_M is singular with a singularity of type A_1. One can show that blowing-up this singular point into a $\mathbb{P}^1$ produces a Kodaira fiber of type I_2. We conclude that there is an isolated fiber in W_M that blows-up into an I_2 fiber.

Now consider the specific point $a \in \mathbb{P}^1$. Then $p_1^{-1}(a) = r$ is a $\mathbb{P}^1$ which intersects t in three points and has a single point of intersection with s. Then, exactly as in the previous section, we see that the fiber $\tilde{p}_1^{-1}(a)$ in W_M consists of two copies of $\mathbb{P}^1$ identified at the four intersection points. Furthermore, since t, s and r are all components of M which intersect transversely, it follows that the two $\mathbb{P}^1$'s must be identified and that each of the intersection points is an isolated singularity of type A_1 in W_M. This is precisely the f_∞ fiber discussed earlier. Recall that it can be resolved into an I_0^*.

To finish the analysis of W_M, note that curve s is a curve of bidegree $(0,1)$ which intersects r. It follows that s can play the role of the curve u in the previous section. Note, however, that s is a component of the branch curve $M = t \cup s \cup r$ and s intersects t and r. It follows that $\pi^{-1}(s)$ is a double $\mathbb{P}^1$ which passes through two isolated A_1 singularities, one at the intersection with $\pi^{-1}(r)$ and one at the intersection with $\pi^{-1}(t)$. Be that as it may, $\pi^{-1}(s)$ is a section of the fibration (5.4), which we will choose to be the zero section.

Putting everything together, we see that W_M in (5.4) is an elliptic fibration with four I_1 fibers. There are two additional degenerate fibers. The first consists of two $\mathbb{P}^1$ lines identified at three points, one of which is an A_1 singularity in W_M, which blows-up to give a I_2 fiber. The second degenerate fiber is a double line with four isolated type A_1 singularities that can be blown-up to give a I_0^* Kodaira fiber.

Having discussed W_M, we use the identification (4.16) that $W_M \cong B/\alpha_B$ to construct the associated space B and involutions α_B. We will denote by $\widehat{W_M}$ the smooth surfaces

$\widehat{B/\alpha_B}$, which are obtained by blowing up each isolated A_1 singularity in B/α_B with a $\mathbb{P}^1$. These surfaces are elliptic fibration over $\mathbb{P}^1$ with four I_1 fibers, an I_2 fiber and one I_0^* Kodaira fiber. Using (2.17) and (4.2) we see that

$$\chi\left(\widehat{W_M}\right) = 12 \tag{5.8}$$

and, hence, $\widehat{W_M} \cong d\mathbb{P}_9$. The singular fibration $W_M \cong B/\alpha_B$ is its Weierstrass model.

Now let us consider the space B. Recall from (4.4) that B is a double cover of B/α_B branched over the fiber f_0 and the four singular points e, e', e'' and e''' on the fiber f_∞. First consider the four I_1 fibers on B/α_B. Since α_B must take fibers of the same Kodaira type to each other in order to be an involution, it follows that there must be eight I_1 fibers on B. Next we discuss the fiber over $p_1(p)$, whose singular point is also an A_1 singularity in W_M and blows up to I_2 in $\widehat{W_M}$. Clearly, $\kappa^{-1}(\tilde{p}^{-1}(p_1(p)))$ are two disjoint copies of this fiber, each containing an A_1 singularity of B. Since B, unlike B/α_B, must be smooth, it is essential that we blow-up each of the two singularities into a $\mathbb{P}^1$. This introduces two new classes of curves, which we denote by n_1 and n_2, beyond those associated with the generic surfaces B discussed previously. Be that as it may, we will continue to denote this blown-up surface and its projection to B/α_B by B and κ respectively. In addition, we will give names, o_1 and o_2, to the proper transforms of the old singular fibers. Recall that the proper transform of the singular curves are their pre-images in the blown-up surface B reduced by the exceptional divisors n_1 and n_2. Both components, the proper transform and the new exceptional divisor, together form a new fiber of Kodaira type I_2. These classes are illustrated in figure 8. Hence, there are two I_2 fibers on B. Now consider the fiber in B/α_B containing the original four isolated A_1 singularities of the surface B. Clearly, the pre-image of this fiber in B must be the unique fiber f_∞. The four singular points in B/α_B arise from the four invariant points e, e', e'', e''' on f_∞ in B. That is, f_∞ is smooth and needs no further resolution.

We conclude that the surfaces B constructed in this section have as degenerate fibers eight I_1 and two I_2 fibers. Note that this configuration satisfies equation (2.20) for $n = 4, m = 2$. Therefore, these surfaces B are, indeed, rational elliptic, that is $B = d\mathbb{P}_9$.

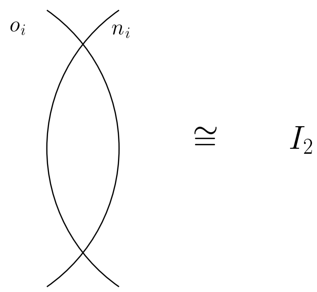

Figure 8: The form of the two reducible I_2 fibers consisting of $o_i \cup n_i$ for $i = 1, 2$.

Finally, let us analyze the zero section of B. Recall that the section $\pi^{-1}(s)$ of B/α_B is composed of two identified $\mathbb{P}^1$ lines with two isolated A_1 singularities in B/α_B, one at the intersection with r and the other at the intersection with t. The pre-image of the curve $\pi^{-1}(r)$ under κ is f_∞. Hence, the A_1 singularity at the intersection with r is simply the invariant point e on f_∞, which is smooth. However, the second A_1 singularity lives on $\pi^{-1}(t)$. Therefore, the pre-image of this point under κ consists of the two isolated A_1 singularities. These are the singularities that are blown-up by n_1 and n_2 to produce the non-singular surface B. Finally, note that, by construction, the zero section e is stable under α_B. It follows that $\kappa^{-1}(\pi^{-1}(s))$, after reducing it by the two exceptional divisors n_1 and n_2, is a single copy of $\mathbb{P}^1$ which intersects n_1 and n_2 and is a section of B. We choose the zero section e of B to be

$$e = \left\{\kappa^{-1}\left(\pi^{-1}(s)\right) \setminus n_1 \cup n_2\right\}. \tag{5.9}$$

We can use the characterization of B/α_B as the double cover of Q branched over the curve M to compute the dimension of its moduli space, as well as the moduli space of B. Recall that $M = t \cup s \cup r$. The curve r has bidegree $(1,0)$ and, hence, it follows from (4.18) that

$$\dim |\mathcal{O}_Q(1,0)| = 1\,. \tag{5.10}$$

Similarly, since curves s and t have bidegree $(0,1)$ and $(1,3)$ respectively, we find from (4.18) that

$$\dim |\mathcal{O}_Q(0,1)| = 1 \tag{5.11}$$

and

$$\dim |\mathcal{O}_Q(1,3)| = 7\,. \tag{5.12}$$

Note that (5.11) is consistent with equation (5.3) and simply tells us that the complex constant b is the modulus of s. As discussed previously, the moduli of $W_M \cong B/\alpha_B$ are the moduli of M reduced by the number of automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ given by (4.21) plus the location of the fiber f_0. That is,

$$\dim \mathcal{M}\left(B/\alpha_B\right) = 1 + 1 + 7 - 6 + 1 = 4\,, \tag{5.13}$$

where $\mathcal{M}(B/\alpha_B)$ is the moduli space of B/α_B. Since surface B is the unique double cover of B/α_B branched over f_0 and the four singular points on f_∞, it follows that B has the same number of moduli as B/α_B, namely

$$\dim \mathcal{M}(B) = 4\,. \tag{5.14}$$

These results have a simple interpretation. Since the curve $t \cup s$ is a degeneration of the generic curve T, the rational elliptic surfaces B obtained here are a subset of the generic surface B constructed from $M = T \cup r$ in the previous section. Specifically, they are the $d\mathbb{P}_9$ surfaces where four of the twelve I_1 fibers have coalesce pairwise to form two I_2 Kodaira fibers. These surfaces form a four parameter subspace of the full five parameter moduli space of surfaces B.

Now, within this restricted four parameter class of surfaces, we will solve for sections ξ that satisfy equation (3.12), that is

$$\alpha_B(\xi) = (-1)_B(\xi)\,. \tag{5.15}$$

To do so, recall from (3.49) and (3.50) that generically solutions of (5.15) come in pairs $(\xi, (-1)_B(\xi))$, with $\xi(0) = e, e', e''$ or e'''. See for example figure 9a. The first thing to notice is that, since these sections map to each other under α_B, it follows that their images after modding out by α_B are identical. That is

$$\kappa(\xi) = \kappa((-1)_B(\xi))\,. \tag{5.16}$$

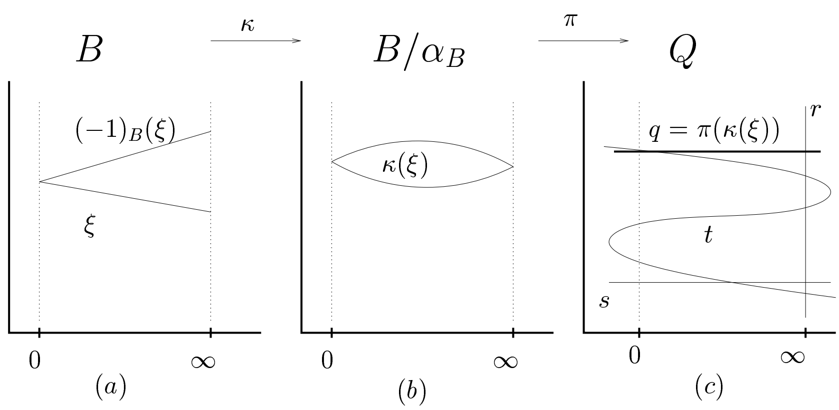

Figure 9: Successive images of a section ξ under the various quotient maps.

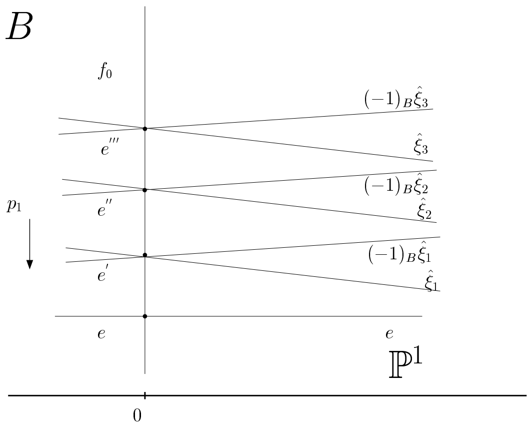

Figure 10: Diagram of B showing the fiber f_0, the four points of order two on f_0 and the seven sections of B satisfying $\alpha_B(\xi) = (-1)_B(\xi)$.

Next, we are interested in the image of one of these sections, call it ξ, under the quotient maps κ and π. Recall that $\xi \cong \mathbb{P}^1$. By construction $\kappa(\xi \cup (-1)_B(\xi))$ is isomorphic to a single $\mathbb{P}^1$. However using the remark above that $\kappa(\xi) = \kappa((-1)_B(\xi))$, we see that $\kappa(\xi) \cong \mathbb{P}^1$ as well. It is clear that $\kappa(\xi)$ intersects each fiber of B/α_B at least once. Noticing that κ identifies two different fibers of B and that, generically, ξ intersects these fibers at different points, we see that actually $\kappa(\xi)$ intersects each fiber of B/α_B twice. Hence $\kappa(\xi)$ is a double section of $B/\alpha_B \cong W_M$. Recalling that $\beta_{\alpha_B} : W_M \to \mathbb{P}^1$, it follows that the image $\kappa(\xi)$ is a double cover of $\mathbb{P}^1$ induced by the restriction of the projection β_{α_B} to $\kappa(\xi)$. Naively, it therefore appears that $\kappa(\xi)$ consists of two copies of $\mathbb{P}^1$. But we have just shown that $\kappa(\xi) \cong \mathbb{P}^1$. These two points of view can be reconciled by noting that the double cover is branched over two points, 0 and ∞, in the base $\mathbb{P}^1$. Therefore, one of the ramification points is in the fiber f_0 and the other in f_∞ of W_B. This situation is indicated in figure 9*b*. It can be shown that $\kappa(\xi)$ is stable under the covering involution for $\pi : W_M \to Q$. Since π projects each fiber of W_M to a unique fiber of Q, the two points of $\kappa(\xi)$ intersecting a fiber in B/α_B get identified under $\pi(\kappa(\xi))$. Therefore, the image of ξ in Q is a section, which we call q. Assuming that the double section $\kappa(\xi)$ is smooth, it can be shown that q has bidegree $(0, 1)$. Henceforth, we will always make that assumption. Furthermore, since the pre-image $\pi^{-1}(q)$ has only one point in the fibers f_0 and f_∞ in B/α_B, q needs to intersect the branch locus of B/α_B at the fibers in Q over 0 and ∞. One such curve q is shown in figure 9*c*.

Clearly, there are three distinct curves $q_i, i = 1, 2, 3$ in Q, each of bidegree $(0, 1)$, which satisfy these conditions. Their pre-image in B are three pairs of sections $(\hat{\xi}_i, (-1)_B(\hat{\xi}_i))$ for $i = 1, 2, 3$ passing through the invariant points e', e'', e''' of f_0 respectively. These sections, as well as the zero section, are shown in figure 10. We conclude that for restricted surfaces B, there are three pairs of sections satisfying (3.12).

6. Surfaces with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ and a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ group of automorphisms

Thus far, we have considered surfaces B that admit involutions $\tau_B = t_\xi \circ \alpha_B$. Are there surfaces B which admit two commuting involutions of this form? This can indeed be achieved. We will show that we have to further restrict to a three dimensional sub-family in the moduli space of B in order to obtain two commuting involutions. Since these surfaces B are the ingredients used to construct Calabi-Yau spaces X with two commuting, freely acting involutions, we will call them surfaces with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group.

To find the constraints on such surfaces, consider two involutions of the form

$$\tau_{Ba} = t_{\xi_a} \circ \alpha_B \tag{6.1}$$

for $a = 1, 2$, where each ξ_a is one of the sections $\hat{\xi_1}, \hat{\xi_2}$ or $\hat{\xi_3}$. Since $\tau_{B1}^2 = \tau_{B2}^2 = id$, it is straightforward to see that these two involutions will commute,

$$\tau_{B1} \circ \tau_{B2} = \tau_{B2} \circ \tau_{B1}\,, \tag{6.2}$$

if and only if

$$(\tau_{B1} \circ \tau_{B2})^2 = \mathrm{id}\,. \tag{6.3}$$

Let us see under what conditions this will be true. Note that

$$\tau_{B1} \circ \tau_{B2} = t_{\xi_1 - \xi_2}\,, \tag{6.4}$$

where we have used $\tau_{B2} = t_{\xi_2} \circ \alpha_B = \alpha_B \circ t_{-\xi_2}$. Hence, it follows from (6.3) and (6.4) that τ_{B1} and τ_{B2} will commute if and only if

$$(t_{\xi_1 - \xi_2})^2 = \mathrm{id}\,. \tag{6.5}$$

Therefore, $\xi_1 - \xi_2$ must be a section of B whose intersection with any fiber f is an invariant point of that fiber under $(-1)_B$. Those points are the points of order two. One possible solution is $\xi_1 - \xi_2 = e$. But then $\xi_1 = \xi_2$ and, hence, $\tau_{B1} = \tau_{B2}$. Therefore, we must find another section that is invariant under $(-1)_B$ which is not the zero section. For the surfaces B studied thus far, there is no such section. However, as we will presently demonstrate, such a section will exist for surfaces B whose quotient B/α_B is the double cover of Q branched over curve M of the form (5.2) with the additional restriction that

$$t = t' \cup i\,, \tag{6.6}$$

where the curve t' has bidegree $(1, 2)$ and the curve i bidegree $(0, 1)$. That is, we henceforth assume

$$M = t' \cup i \cup s \cup r\,. \tag{6.7}$$

As mentioned at the end of section 3, a purely translational involution of the form (6.5) can lift to freely acting involution on X.

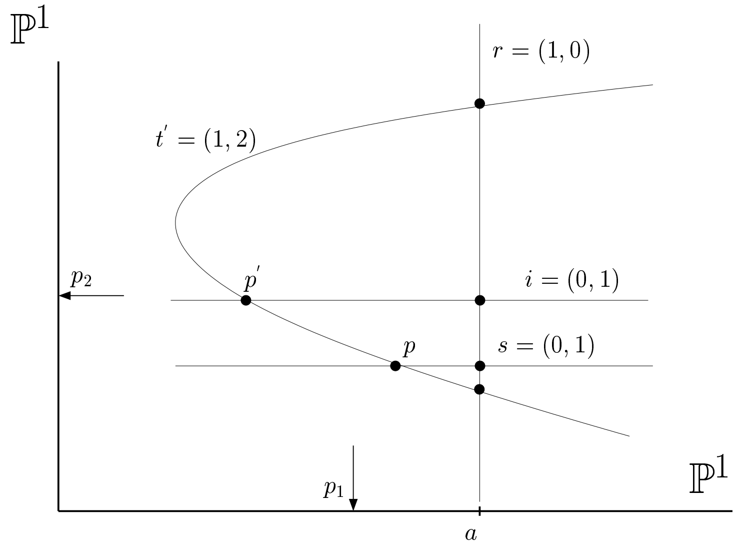

Figure 11: A schematic representation of curve $M = t^{'} \cup i \cup s \cup r$ in $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

Recall that z_i are the complex coordinates of $\mathbb{P}^1_i$ for $i = 1, 2$. In terms of these coordinates, the curve $i \subset Q$ of bidegree $(0, 1)$ must satisfy the linear equation

$$-c + z_2 = 0, \tag{6.8}$$

where c is a complex coefficient. Hence, $z_2 = c$. We refer the reader to figure 11 for a pictorial representation of curve i, as well as the curves s and r of bidegree $(0, 1)$ and $(1, 0)$ respectively given in (4.7) and (5.3). The curve $t^{'}$, on the other hand, must satisfy an equation that is linear in z_1 and is of degree two in z_2. Note that s and i each intersect the curve $t^{'}$ at a single point, which we denote by p and $p^{'}$ respectively. Clearly, the curve r located at fixed $z_1 = a$ will intersect each of s and i at a single point, whereas r will generically intersect $t^{'}$ at two points. As before, we will assume that these intersections are transversal. Again, we refer the reader to figure 11 for an illustration of these remarks.

Having specified the curve M in (6.7), we now analyze the double cover W_M of Q branched over M. As discussed previously, W_M is a fibration, $\tilde{p}_1 : W_M \to \mathbb{P}^1_1$, over $\mathbb{P}^1$ where $\tilde{p}_1$ is defined by (4.9). We begin by considering a generic point $x \in \mathbb{P}^1$, where $x \neq a, p_1(p)$ or $p_1(p^{'})$. Then $p_1^{-1}(x)$ is a $\mathbb{P}^1$ that intersects curve $T = t^{'} \cup i \cup s$ at four points. Therefore, $\tilde{p}_1^{-1}(x)$ is a double cover of $\mathbb{P}^1$ branched at those four points. It follows that $\tilde{p}_1^{-1}(x)$ is a torus and, hence, the fibration (5.4) is a torus fibration.

To explore the possible degenerate fibers of W_M, we have to analyze more closely the structure of the curves t', i and s in Q. Consider, for a moment, the projection $p_2 : Q \to \mathbb{P}^1_2$. Clearly $p_2 : t^{'} \to \mathbb{P}^1_2$ is a degree one cover and, hence, $t^{'} \cong \mathbb{P}^1$. Now consider $p_1 : Q \to \mathbb{P}^1_1$. Under p_1, $t^{'}$ is clearly a two-fold cover of $\mathbb{P}^1$. As already mentioned several times in this paper, the condition for a double cover of $\mathbb{P}^1$ to be isomorphic to $\mathbb{P}^1$ implies that the cover has two branch points. Let b_i, $i = 1, 2$ denote these points in $\mathbb{P}^1$. Clearly, for each b_i $\tilde{p}_1^{-1}(b_i)$ is a double cover of $\mathbb{P}^1$ branched at three, not four, points, where one of these points is in $t^{'}$, one in i and one in s. Similarly to the discussion in the previous section, the fiber $\tilde{p}_1^{-1}(b_i)$ will be singular of Kodaira type I_1. Since the branch curve is smooth at this point, this is not a singularity in W_M and need not be blown-up. We conclude that there are two I_1 fibers that arise in this manner.

Are there other degenerate fibers? Let us consider the intersection point p' of the curves t' and i. This projects to a point $p_1(p')$ in $\mathbb{P}^1_1$ and lies in the fiber $p_1^{-1}(p_1(p'))$ in Q. Note from figure 11 that $p_1^{-1}(p_1(p'))$ intersects each of t' and s at a single point and both t' and i simultaneously at p'. It follows that $\tilde{p}_1^{-1}(p_1(p'))$ is a fiber in W_M which is a double cover of $\mathbb{P}^1$ branched at three points. Furthermore, this fiber must be singular at the point p'. In addition, since, by assumption, t' and i intersect, it follows that the branch locus of W_M is not smooth at p' and, hence, the singularity is of type A_1 in the space W_M. Therefore, as discussed previously, blowing-up the singular point into a $\mathbb{P}^1$ produces a Kodaira fiber of type I_2 in $\widehat{W_M}$. Exactly the same comments can be made concerning the intersection point p of the curves t' and s. We conclude that there are two isolated fibers in W_M, each of which blows-up into an I_2 fiber. Now consider the specific point $a \in \mathbb{P}^1_1$. Then $p_1^{-1}(a) = r$ is a $\mathbb{P}^1$ which intersects t' in two points and has a single intersection with each of i and s. Then exactly as in the previous cases, we see that fiber $\tilde{p}_1^{-1}(a)$ in W_M consists of two copies of $\mathbb{P}^1$ identified at the four intersection points. Since t', i, s and r are all components of M which intersect transversely, it follows that the two $\mathbb{P}^1$ copies must be identified and that each of the intersection points is an isolated singularity of type A_1 in W_M. This is precisely the f_∞ fiber discussed earlier that can be resolved into I_0^*.

To finish the analysis of W_M, note that curve s has exactly the same properties as it had in the restricted $\mathbb{Z}_2$ case in the previous section. Therefore, $\pi^{-1}(s)$ is a section of the fibration (5.4), which we choose to be the zero section. Now consider the curve i of bidegree $(0,1)$, which is new to this analysis. It is not hard to see that it has exactly the same properties as the curve s. That is, $\pi^{-1}(i)$ is two identified $\mathbb{P}^1$ lines with two isolated A_1 singularities, one at the intersection with r and one at the intersection with t'. Since both intersections are transverse, these are singularities in the space W_M. Be that as it may, $\pi^{-1}(i)$ is a section of the fibration (5.4) which is distinct from $\pi^{-1}(s)$.

Putting everything together, we see that W_M in (5.4) is now an elliptic fibration with two I_1 fibers and two fibers with an A_1 singularity in W_M, which can be resolved by blowing-up into I_2 fibers in $\widehat{W_M}$. In addition, there is another degenerate fiber which is a double line containing four isolated A_1 singularities in W_M, that can be blown-up into an I_0^* Kodaira fiber. Note that the surface $\widehat{W_M}$, obtained by blowing up each isolated A_1 singularity in W_M with a $\mathbb{P}^1$, is an elliptic fibration over $\mathbb{P}^1$ with two I_1 fibers, two I_2 fibers and one I_0^* Kodaira fiber. Using (2.17) and (4.2), we see that

$$\chi\left(\widehat{W_M}\right) = 2\chi(I_1) + 2\chi(I_2) + \chi(I_0^*) = 12 \tag{6.9}$$

and, hence, $\widehat{W_M} = d\mathbb{P}_9$. The singular fibration W_M is its Weierstrass model.

Having discussed W_M and its resolution $\widehat{W_M}$, we use the identification (4.16) that $W_M \cong B/\alpha_B$ to construct the associated spaces B and involutions α_B. Recall from (4.4) that B is the double cover of B/α_B branched over the fiber f_0 and the four singular points e, e', e'' and e''' on the fiber f_∞. First, consider the two I_1 fibers in B/α_B. Since α_B must map Kodaira fibers of the same type into each other, it follows that there must be four I_1 fibers in B. Next, we discuss the fiber over $p_1(p)$ which contains an A_1 singularity of W_M and blows-up into a I_2 in $\widehat{W_M}$. As discussed above, $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p)))$ are two disjoint copies of this fiber, each containing an A_1 singularity of B. To make B smooth, we must blow-up each of these singularities into a $\mathbb{P}^1$, thus introducing two new classes of curves, which we denote by n_1 and n_2. We give the names o_1 and o_2 to the associated proper transforms. Each of the pairs (n_i, o_i), $i = 1, 2$ is an I_2 Kodaira fiber as illustrated in figure 8. Similarly, consider the fiber over $p_1(p^{'})$. Clearly, this behaves in exactly the same way as the pre-image of $p_1(p)$. That is, $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p)))$ is two disjoint fibers each with an A_1 singularity in B. To make B smooth, we must also blow up these two singularities. This introduces two additional classes of curves, which we denote by n_3 and n_4. We also denote by o_3 and o_4 the associated proper transforms. Again, each of the pairs (n_i, o_i), $i = 3, 4$ is an I_2 Kodaira fiber. Therefore, there are, in total, four I_2 fibers in B. Now consider the fiber in B/α_B with four isolated A_1 singularities. Clearly, the pre-image of this fiber in B must be the unique fiber f_∞. The four singular points in the B/α_B fiber arise from the four invariant points $e, e^{'}, e^{''}$ and $e^{'''}$ on f_∞ in B. That is, f_∞ is smooth and needs no further resolution. We conclude that the surfaces B constructed in this section have as degenerate fibers four I_1 and four I_2 fibers. Note that this configuration satisfies equation (2.20) for $n = 2, m = 4$. Therefore, these surfaces B are, indeed, rational elliptic, that is, $B = d\mathbb{P}_9$.

Let us analyze $\kappa^{-1}\pi^{-1}(s)$ and $\kappa^{-1}\pi^{-1}(i)$ of B. First consider $\kappa^{-1}\pi^{-1}(s)$. Exactly as in the previous case, $\kappa^{-1}\pi^{-1}(s)$ reduced by the exceptional divisors n_1 and n_2 is a section of B, which is single copy of $\mathbb{P}^1$ which intersects n_1 and n_2. We define $\kappa^{-1}\pi^{-1}(s)$ minus the exceptional divisors to be the zero section e of B. In the present case, there is a second bidegree $(0, 1)$ curve on Q, namely, i. Clearly, the properties of $\pi^{-1}(i)$ are identical to those of $\pi^{-1}(s)$ with the exception that the second A_1 singularity in $\pi^{-1}(i)$ lies at the point $p^{'}$ of the $\tilde{p}_1^{-1}(p_1(p^{'}))$ fiber instead of at p in $\tilde{p}_1^{-1}(p_1(p))$. Note that the pre-image of $p^{'}$ under κ consists of the two isolated A_1 singularities that are blown-up by n_3 and n_4. A key property of the curve i is that, by construction, it must coincide with one of the curves q_i defined at the end of the previous section. The reason is that it is a bidegree $(0, 1)$ curve distinct from s which passes through an intersection point of $t = t^{'} \cup i$ with f_0. However, unlike the generic curves q_i, this specific curve, since it is a component of T, also passes through one of the invariant A_1 singular points on r. It is clear, therefore, that the proper transform of that curve in B, which we denote by e_6, is a section of B satisfying

$$e_6 = (-1)_B(e_6)\,. \tag{6.10}$$

That is, e_6 is a section of B with the property that $e_6|_f$ is an invariant point of each fiber f under $(-1)_B$. To fix notation, we will label the points of order two so that

$$e_6|_f = e^{'}\,. \tag{6.11}$$

The designation of the proper transform of $\kappa^{-1}\pi^{-1}(i)$ as e_6 arises as follows. Recall from section 2 that there are nine sections $e_i, i = 1, \ldots, 9$ of B with fibration (5.4), corresponding to the nine $\mathbb{P}^1$ blow-ups of the $d\mathbb{P}_9$. We have already designated e_9 corresponding to the proper transform of $\kappa^{-1}\pi^{-1}(s)$ as the zero section in (2.12). In [15], we will show that e_6 can indeed be identified with the section associated to $\kappa^{-1}\pi^{-1}(i)$. In anticipation of that result, we here denote e_6 as the proper transform of $\kappa^{-1}\pi^{-1}(i)$. To conclude, we see that $\kappa^{-1}\pi^{-1}(i)$ minus the exceptional divisors n_3 and n_4 is a section of B, which is a single copy of $\mathbb{P}^1$ intersecting the exceptional divisors n_3 and n_4. Having identified i with one of the curves q_i, let us arbitrary set $i = q_3$. It follows that, in addition to e and e_6, there are two pairs of sections $(\hat{\xi}_i, (-1)_B(\hat{\xi}_i))$ for $i = 1, 2$ that satisfy condition (5.15). We can choose them so that $\hat{\xi}_1(0) = e^{''}$ and $\hat{\xi}_2(0) = e^{'''}$ respectively.

It is now clear that these restricted B surfaces admit a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. To see this, consider

$$\tau_{B1} = t_\xi \circ \alpha_B\,, \tag{6.12}$$

where section ξ is chosen to be either $\hat{\xi}_1$ or $\hat{\xi}_2$ and

$$\tau_{B2} = t_{\xi - e_6} \circ \alpha_B\,. \tag{6.13}$$

Note that both τ_{B1} and τ_{B2} are involutions since ξ and $\xi - e_6$ satisfy condition (5.15). Furthermore,

$$\tau_{B1} \circ \tau_{B2} = t_{e_6}\,. \tag{6.14}$$

Since e_6 is a point of order two on each fiber, it follows that condition (6.3) is satisfied and, hence, by the previous discussion

$$\tau_{B1} \circ \tau_{B2} = \tau_{B2} \circ \tau_{B1}\,. \tag{6.15}$$

Therefore, B admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action.

We can use the characterization of B/α_B as the double cover of Q branched over curve M to compute the dimension of its moduli space, as well as the moduli space of B. Recall that for these restricted surfaces $M = t^{'} \cup i \cup s \cup r$. The curve r has bidegree $(1, 0)$ and, hence, it follows from (4.18) that

$$\dim |\mathcal{O}_Q(1, 0)| = 1\,. \tag{6.16}$$

Similarly, the curves s and i have bidegree $(0, 1)$. We then find, using (4.18), that

$$\dim |\mathcal{O}_Q(0, 1)| = 1 \tag{6.17}$$

for each curve. Finally, since the curve $t^{'}$ has bidegree $(1, 2)$, equation (4.18) implies that

$$\dim |\mathcal{O}_Q(1, 2)| = 5\,. \tag{6.18}$$

As discussed previously, the number of the moduli of $W_M \cong B/\alpha_B$ is the number of the moduli of M reduced by the dimension of the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ given in (4.21) and enhanced by the location parameter of fiber f_0. That is,

$$\dim \mathcal{M}\,(B/\alpha_B) = 1 + 1 + 1 + 5 - 6 + 1 = 3\,. \tag{6.19}$$

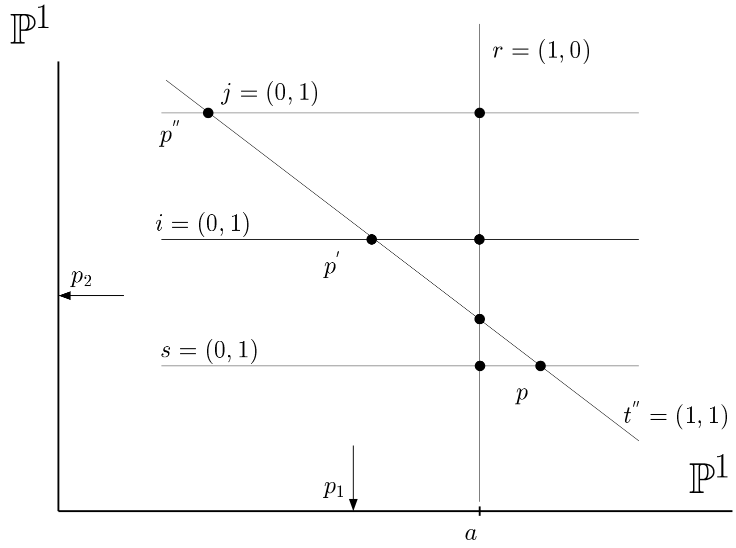

Figure 12: A schematic representation of curve $M = t'' \cup j \cup i \cup s \cup r$ in $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

Since surface B is the double cover of B/α_B branched over f_0 and the four singular points in f_∞, it follows that B has the same number of moduli as B/α_B, namely

$$\dim \mathcal{M}(B) = 3\,. \tag{6.20}$$

These results have a simple interpretation. Since the curve $t' \cup i \cup s$ is a degeneration of the generic curve T, the rational elliptic surfaces B obtained here are a subset of the surfaces B constructed from $M = t \cup s \cup r$ and $M = T \cup r$. Specifically, they are $d\mathbb{P}_9$ surfaces where eight of the twelve I_1 fibers coalesce together in pairs to form four I_2 Kodaira fibers. These surfaces form a three parameter subspace of the full five parameter moduli space of surfaces B admitting an involution α_B. Finally, we note that, in addition to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action, the restricted surfaces discussed in this section contain four classes of curves, namely n_1, n_2, n_3 and n_4, that are not present for a generic surface.

However, we are not quite finished. Our aim is to construct rational elliptic surfaces giving rise to Calabi-Yau threefolds of the form $X = B \times_{\mathbb{P}^1} B'$ admitting two freely acting commuting involutions. Furthermore, X must admit stable, holomorphic vector bundles which can be used to obtain standard-like models in string theory compactifications. In order to obtain standard-like models, as we will show in [15], it is necessary to have invariant classes in $H_2(B, \mathbb{Z})$ which consist of fiber components, but do not contain a whole fiber. To find such invariant classes, we find it necessary to restrict the surfaces B once more. As we will see, surfaces of this highly restricted type admit three commuting involutions τ_{B1}, τ_{B2} and τ_{B3}, so that we call them surfaces with a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group.

Since the techniques involved have all been discussed above, we will only present the results. Let us split the curve t' as

$$t' = t'' \cup j\,, \tag{6.21}$$

where curve t'' has bidegree $(1,1)$ and curve j has bidegree $(0,1)$. That is, we henceforth assume

$$M = t^{''} \cup j \cup i \cup s \cup r\,. \tag{6.22}$$

We refer the reader to figure 12 for a pictorial representation of the curve M in Q. The points of intersection of $t^{''}$ with s, i and j are denoted by $p, p^{'}$ and $p^{''}$ respectively. Now consider the double cover W_M of Q branched over M. This is a fibration over $\mathbb{P}^1$ with projection map $\tilde{p}_1$ specified in (5.4). For $x \in \mathbb{P}^1$ where $x \neq a, p_1(p), p_1(p^{'})$ or $p_1(p^{''})$ one can show, exactly as in the previous cases, that

$$\tilde{p}_1^{-1}(x) = T^2\,, \tag{6.23}$$

where T^2 is a smooth torus. Hence, (5.4) is a torus fibration. However, as above, we expect there to be degenerate fibers. Using the fact that $t^{''} \cong \mathbb{P}^1$ and $p_1|_{t^{''}} : t^{''} \to \mathbb{P}^1$ is an isomorphism, it follows that there are no I_1 fibers on W_M. Now, exactly as above, we see that $\tilde{p}_1^{-1}(p_1(p))$ is a fiber of W_M which is a double cover of $\mathbb{P}^1$ branched at three points. One of these points, p, is an A_1 singularity of W_M. This fiber can be blown-up at p to produce an I_2 Kodaira fiber. The fibers $\tilde{p}_1^{-1}(p_1(p^{'}))$ and $\tilde{p}_1^{-1}(p_1(p^{''}))$ each have the same properties. That is, there are three isolated singular fibers which blow-up to I_2. Finally, there is the fiber $\tilde{p}_1^{-1}(a)$ with four A_1 singularities in W_M. This is the f_∞ fiber, which can be resolved into I_0^*.

Note that $\widehat{W_B}$, obtained by blowing-up all A_1 singularities in W_M, is an elliptic fibration over $\mathbb{P}^1$ with no I_1 fibers, three I_2 fibers and one I_0^* fiber. Using (2.17) and (4.2), we see that $\chi(\widehat{W_M}) = 12$ and, hence, $\widehat{W_M} \cong d\mathbb{P}_9$. The singular fibration W_M is its Weierstrass model. As previously, $\pi^{-1}(s)$ is the zero section of the fibration (5.4). Clearly, each of $\pi^{-1}(i)$ and $\pi^{-1}(j)$ have the same properties. That is, each consists of a double $\mathbb{P}^1$ containing two isolated A_1 singularities of W_M, one at the intersection of $\pi^{-1}(r)$ and one at the intersection with $\pi^{-1}(t^{''})$. Both are sections of W_M, distinct from each other and from the zero section.

We now identify $W_M \cong B/\alpha_B$ and consider space B. Recall from (4.4) that B is the double cover of B/α_B branched over the fiber f_0 and the four singular points $e, e^{'}, e^{''}$ and $e^{'''}$ on the fiber f_0. Since there are no I_1 fibers on B/α_B, there will be none in B. Recall that $\kappa^{-1}(\tilde{p}^{-1}(p_1(p)))$ are two separated copies of this fiber, each with an A_1 singularity in B. These must be blown-up, thus introducing two new classes n_1 and n_2. The associated proper transforms are denoted o_1 and o_2. Each of these blown-up fibers is of Kodaira type I_2. Clearly, $\kappa^{-1}(\tilde{p}^{-1}(p_1(p^{'})))$ and $\kappa^{-1}(\tilde{p}^{-1}(p_1(p^{''})))$ have the same properties. That is, each must have its pair of singularities blown-up, thus introducing additional curve classes n_3, n_4 and n_5, n_6, with o_3, o_4 and o_5, o_6 denoting the proper transforms. This yields four more I_2 fibers in B, for a total of six.

Finally, consider the fiber in B/α_B with four isolated A_1 singularities. Clearly, the pre-image of this fiber in B is the non-singular fiber f_∞. We conclude that the surfaces B, so constructed, have six I_2 degenerate fibers. Note that this configuration satisfies equation (2.20) for $n = 0, m = 6$. Therefore, these surfaces are, indeed, rational elliptic, that is $B = d\mathbb{P}_9$.

Let us analyze the sections associated with $\kappa^{-1}\pi^{-1}(s)$, $\kappa^{-1}\pi^{-1}(i)$ and $\kappa^{-1}\pi^{-1}(j)$. As above, $\kappa^{-1}\pi^{-1}(s)$ reduced by the exceptional divisors n_1 and n_2 is a section of B, which is a $\mathbb{P}^1$ which intersects n_1 and n_2. We define as in (5.9)

$$e = \{\kappa^{-1}\pi^{-1}(s) \setminus n_1 \cup n_2\} \tag{6.24}$$

to be the zero section of B. Similarly, $\kappa^{-1}\pi^{-1}(i)$ reduced by the exceptional divisors is a $\mathbb{P}^1$ intersecting the pair of I_2 fibers $\kappa^{-1}(\tilde{p}^{-1}(p_1(p^{'})))$ in n_3 and n_4. As discussed above,

$$e_6 = \{\kappa^{-1}\pi^{-1}(i) \setminus n_3 \cup n_4\} \tag{6.25}$$

is the section of B which passes through the invariant point $e^{'}$ on each fiber. As will be shown elsewhere [15], this section can be identified with e_6 defined in section 2. Clearly, by construction, $\kappa^{-1}\pi^{-1}(j)$ has similar properties. That is, $\kappa^{-1}\pi^{-1}(j)$ reduced by the exceptional divisors is a $\mathbb{P}^1$ section of B intersecting the pair of I_2 fibers on $\kappa^{-1}(\tilde{p}^{-1}(p_1(p^{''})))$ in n_5 and n_6. This section has the property that it passes through a point of order two distinct from e and $e^{'}$ on each fiber. To fix notation we denote this to be $e^{''}$. It will be shown in [15] that this section can be identified with the section e_4 defined in section 2. In anticipation of this, we denote

$$e_4 = \{\kappa^{-1}\pi^{-1}(j) \setminus n_5 \cup n_6\}\,. \tag{6.26}$$

As discussed above, curve i, by construction, must be identified with one of the curves q_i. As before, we choose $i = q_3$. Since i is in the branch locus of M, its pre-image in B determines a unique section, e_6. This is consistent with the fact (6.10) that $e_6 = (-1)_B(e_6)$. In exactly the same way, the curve j can be identified with one of the two remaining curves q_i, which we arbitrarily choose as $j = q_2$. Since j is a component of M, its pre-image in B determines a unique section e_4. This leaves one more curve, q_1. Hence, there exists exactly one pair of sections $(\xi, (-1)_B(\xi))$ fulfilling condition (3.12). However, in this restricted case, there is an important caveat. Note that the section $e_4 + e_6$ is also a section of B, which intersects each fiber at a point of order two. It can be shown that this section is determined by the pre-image of $t^{''}$ in B and intersects all of the six new classes $n_i, i = 1, \ldots, 6$. However, even through both sections ξ and e_4+e_6 pass through the invariant point $e^{'''}$ at the fiber f_0, they are not identical. Therefore, the section e_4+e_6 is not of the type discussed previously in this paper. It arises in this very restricted surface B, due to the existence of the $t^{''}$ curve in the branch locus of bidegree $(1,1)$.

It is now straightforward to show that these restricted surfaces B admit a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ action. To see this, define

$$\tau_{B1} = t_\xi \circ \alpha_B\,, \qquad \tau_{B2} = t_{\xi - e_4} \circ \alpha_B\,, \qquad \tau_{B3} = t_{\xi - e_6} \circ \alpha_B\,. \tag{6.27}$$

Clearly $\tau_{B1}^2 = \tau_{B2}^2 = \tau_{B3}^2 = id$, since $\xi, \xi - e_4$ and $\xi - e_6$ fulfill condition (3.12). Furthermore,

$$\tau_{B1} \circ \tau_{B2} = t_{e_6}\,, \qquad \tau_{B1} \circ \tau_{B3} = t_{e_4}\,, \qquad \tau_{B2} \circ \tau_{B3} = t_{e_4 - e_6}\,. \tag{6.28}$$

Note that since e_6 is a section of order two, $e_4 - e_6$ is equal to the section $e_4 + e_6$ discussed above. Since each of these quantities squares to the identity map, it follows from condition (6.3) that

$$\tau_{Bi} \circ \tau_{Bj} = \tau_{Bj} \circ \tau_{Bi} \tag{6.29}$$

for $i, j = 1, 2, 3$. Therefore, B admits a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ action.

Notice that for these surfaces B one could take the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by the translations t_{e_6} and t_{e_4}. Recalling the remarks of the end of section 3, these involutions could be lifted to freely acting involutions on the Calabi-Yau threefold X. However, it would not be possible to obtain invariant classes consisting of fiber components only. Hence, they are very unlikely to admit vector bundles which allow standard model-like string compactifications and, hence, we will not consider them further.

Finally, recall that for these restricted surfaces $M = t^{''} \cup j \cup i \cup s \cup r$, where r has bidegree $(1,0)$, the curves s, i and j have bidegree $(0,1)$ and $t^{''}$ has bidegree $(1,1)$. Then, using (6.16), (6.17) and the fact that (4.18) implies

$$\dim |\mathcal{O}_Q(1,1)| = 3\,, \tag{6.30}$$

we find that

$$\dim \mathcal{M}\left(B/\alpha_B\right) = 1+1+1+1+3-6+1 = 2\,, \tag{6.31}$$

where we have subtracted off the number of automorphisms (4.21) of $\mathbb{P}^1 \times \mathbb{P}^1$ and added the location modulus of fiber f_0. As discussed above, it follows that

$$\dim \mathcal{M}(B) = 2\,. \tag{6.32}$$

Since $t^{''} \cup j \cup i \cup s \cup r$ is a degeneration of the generic curve T, the rational elliptic surfaces B obtained here are a subset of the surfaces previously constructed. Specifically, they are $d\mathbb{P}_9$ surfaces where all twelve I_1 fibers have coalesced in pairs to form six I_2 Kodaira fibers. These surfaces form a two parameter subspace of the full five parameter moduli space of surfaces B that admit involutions α_B. We note that, in addition to the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ action, these restricted surfaces contain six classes of curves, namely n_i for $i = 1, \ldots, 6$, that are not present in the generic fibration.

7. The Calabi-Yau threefolds X and Z

Having constructed the rational elliptic surfaces B that admit several commuting involutions, we are finally in a position to construct torus-fibered Calabi-Yau threefolds Z with fundamental group

$$\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2\,. \tag{7.1}$$

To do so, we construct elliptically fibered Calabi-Yau spaces X as the fiber product over $\mathbb{P}^1$ of two $d\mathbb{P}_9$ surfaces, B and $B^{'}$ [9]. Choosing each surface to admit several commuting involutions, we will lift these to two, commuting, freely acting involutions on X. These generate the involution group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The quotient

$$Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2) \tag{7.2}$$

must be a smooth manifold with fundamental group $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2$. We will then prove that, in addition, Z is a torus-fibered Calabi-Yau space. Notice, however, that the quotient Z does not admit any section, and, hence it is not an elliptic fibration.

The fiber product of two rational elliptic surfaces B and $B^{'}$

$$\beta : B \to \mathbb{P}^1 \,, \qquad \beta^{'} : B^{'} \to \mathbb{P}^{1^{'}} \tag{7.3}$$

is defined as

$$B \times_{\mathbb{P}^1} B^{'} = \left\{ \left(b, b^{'}\right) \in B \times B^{'} | \beta(b) = \beta^{'}\left(b^{'}\right) \right\}, \tag{7.4}$$

where we need to make an identification of $\mathbb{P}^1$ and $\mathbb{P}^{1^{'}}$ discussed in more detail below. The manifold $B \times B^{'}$ has complex dimension 4. Clearly, however, the constraint $\beta(b) = \beta^{'}(b^{'})$ reduces the dimension by one. That is,

$$\dim_{\mathbb{C}} \left(B \times_{\mathbb{P}^1} B^{'}\right) = 3\,. \tag{7.5}$$

Now let $\pi : B \times_{\mathbb{P}^1} B^{'} \to B^{'}$ and $\pi^{'} : B \times_{\mathbb{P}^1} B^{'} \to B$ be the natural mappings defined by

$$\pi\left(b, b^{'}\right) = b^{'}\,, \qquad \pi^{'}\left(b, b^{'}\right) = b\,. \tag{7.6}$$

Then, we obtain the commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{7.7}$$

and, hence, one can define a map $\rho : B \times_{\mathbb{P}^1} B^{'} \to \mathbb{P}^1$ as

$$\rho = \beta^{'} \circ \pi = \beta \circ \pi^{'}\,. \tag{7.8}$$

Let us focus on $\pi : B \times_{\mathbb{P}^1} B^{'} \to B^{'}$. Choose $b^{'} \in B^{'}$ and denote $\beta^{'}(b^{'}) = p$. Then the fiber $\pi^{-1}(b^{'})$ in X given by

$$\pi^{-1}\left(b^{'}\right) = \left\{ \left(b, b^{'}\right) \in B \times_{\mathbb{P}^1} B^{'} | b \in \beta^{-1}(p) \right\}, \tag{7.9}$$

is isomorphic to the fiber in B. That is, $\pi^{-1}(b^{'})$ is either a torus or a degeneration thereof. It follows that

$$\pi : B \times_{\mathbb{P}^1} B^{'} \to B^{'} \tag{7.10}$$

is a torus fibration over base $B^{'}$. Of course, the same is true for $\pi^{'} : B \times_{\mathbb{P}^1} B^{'} \to B$. However, following the conventions in previous literature, we will consider $B^{'}$ as the base of the $B \times_{\mathbb{P}^1} B^{'}$ fibration. This explains the, somewhat perverse, notation π as the projection map in (7.10). Now, recall that $e : \mathbb{P}^1 \to B$ is the zero section of the fibration $\beta : B \to \mathbb{P}^1$. It follows from this and (7.9) that mapping $\sigma : B^{'} \to B \times_{\mathbb{P}^1} B^{'}$, defined by its image

$$\sigma = e \times_{\mathbb{P}^1} B^{'}\,, \tag{7.11}$$

is a section of fibration (7.10). We will choose σ as the zero section. Therefore, (7.10) is an elliptic fibration. We conclude that $B \times_{\mathbb{P}^1} B^{'}$ is a threefold that is elliptically fibered over base surface $B^{'}$ with projection map π.

The definition of X seems, at first, rather contrived. However, as we now show, every elliptic Calabi-Yau threefold X fibered over a $d\mathbb{P}_9$ base is, indeed, a fiber product over $\mathbb{P}^1$ of two rational elliptic surfaces. Let X be an elliptic fibration

$$\pi : X \to B^{'} , \tag{7.12}$$

where $B^{'}$ is isomorphic to $d\mathbb{P}_9$. By definition, a generic fiber of π is a torus and there must exist a zero section $\sigma : B^{'} \to X$. Recall from section 2 that $B^{'}$ itself is an elliptic fibration, that is, $\beta^{'} : B^{'} \to \mathbb{P}^1$.

Let us describe the Weierstrass model W_X of X. To do so, recall the construction of the Weierstrass model W_B of an elliptic fibered surface B given in section 2. W_B is cut out of $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_e(-2e) \oplus \mathcal{O}_e(-3e))$ by a global section of $\mathcal{O}_P(3) \otimes p^*\mathcal{O}_e(-e)^6$. That is

$$\begin{array}{ccc} W_B & \subset & \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_e(-2e) \oplus \mathcal{O}_e(-3e)) = P \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array} \quad . \tag{7.13}$$

Note that $\mathcal{O}_e(-ne) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ for any integer n. We argued that, for generic rational elliptic surfaces B, $W_B \cong B$. In analogy to this, the Weierstrass model W_X of X is cut out of $P^{'} = \mathbb{P}(\mathcal{O}_{B^{'}} \oplus \mathcal{O}_\sigma(-2\sigma) \oplus \mathcal{O}_\sigma(-3\sigma))$ by a global section of $\mathcal{O}_{P^{'}}(3) \otimes p^{'*}\mathcal{O}_\sigma(-\sigma)^6$. The associated diagram is

$$\begin{array}{ccc} W_X & \subset & \mathbb{P}(\mathcal{O}_{B^{'}} \oplus \mathcal{O}_\sigma(-2\sigma) \oplus \mathcal{O}_\sigma(-3\sigma)) = P^{'} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p^{'}} \\ B^{'} & = & B^{'} \end{array} \quad . \tag{7.14}$$

Again, for generic Calabi-Yau threefolds X, $W_X \cong X$.

Since for Calabi-Yau spaces the canonical bundle is trivial, the adjunction formula on X allows us to re-express $P^{'}$ in terms of a projective bundle over $\mathbb{P}^1$. To see this, recall that the adjunction formula states

$$K_X|_D = K_D \otimes \mathcal{O}_D(D) , \tag{7.15}$$

where $\mathcal{O}_D(D)$ is the normal bundle for a divisor $D \subset X$, K_D the canonical bundle of D and $K_X|_D$ the canonical bundle of X restricted to D. Since the canonical bundle of X must be trivial, it follows that $K_X|_D = \mathcal{O}_D$. Furthermore, choosing $D = \sigma$ we find

$$\mathcal{O}_\sigma(\sigma) = K_\sigma . \tag{7.16}$$

Using the fact that the section σ is isomorphic to the base $B^{'}$, it follows that $K_\sigma \cong K_{B^{'}}$. Therefore, $\mathcal{O}_\sigma(-n\sigma) = K_{B^{'}}^{-n}$ for any integer n. Furthermore, we also have shown in section 2 that, for a rational elliptic surface $B^{'}$, the canonical bundle is given as $K_{B^{'}} = \mathcal{O}_{B^{'}}(-f^{'}) = \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(-1)$, where $f^{'}$ denotes the fiber class of $B^{'}$ and $\beta^{'}$ the projection. Summarizing, we find that

$$\mathcal{O}_{\sigma}(-n\sigma) = \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(n)\,. \tag{7.17}$$

Using this result, we see that $P^{'}$ can be written as $P^{'} = \mathbb{P}(\beta^{'*}\mathcal{O}_{\mathbb{P}^1} \oplus \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(2) \oplus \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(3))$. That is,

$$\begin{tikzcd} W_X \arrow[d] & \subset & \mathbb{P}(\beta^{'*}\mathcal{O}_{\mathbb{P}^1} \oplus \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(2) \oplus \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(3)) = P^{'} \arrow[d, "p^{'}"] \\ B^{'} & = & B^{'} \end{tikzcd} \tag{7.18}$$

and the Weierstrass model W_X of X is cut out of $P^{'}$ by a global section of $\mathcal{O}_{P^{'}}(3) \otimes p^{'*} \circ \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(6)$.

Observe that $P^{'}$ is the projectivization of a rank three vector bundle on $B^{'}$, which is a pull-back of a vector bundle on $\mathbb{P}^1$. For such bundles, there is a natural projection from the pulled back bundle to the original bundle. We will call this projection λ and obtain

$$\begin{tikzcd} P^{'} = \mathbb{P}(\beta^{'*}\mathcal{O}_{\mathbb{P}^1} \oplus \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(2) \oplus \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(3)) \arrow[r, "\lambda"] \arrow[d, "p^{'}"] & \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)) \arrow[d, "p"] \\ B^{'} \arrow[r, "\beta^{'}"] & \mathbb{P}^1 \end{tikzcd} \quad . \tag{7.19}$$

Recall that W_X is the zero locus of a global section of $\mathcal{O}_{P^{'}}(3) \otimes p^{'*} \circ \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(6)$ in $P^{'}$. What is the image of W_B under λ ? Clearly, it is the zero locus of a section of the bundle $\lambda_*(\mathcal{O}_P(3) \otimes p^{'*} \circ \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(6))$ on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$. It can be shown that this push-forward bundle is $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))}(3) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(6)$. Hence, the image of W_X under λ is a divisor given by the zero locus of a section of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))}(3) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(6)$. But as discussed in section 2, such a divisor corresponds to a Weierstrass model W_B of a rational elliptic surface. Since W_B is generically isomorphic to the surface itself, we will denote it by B. In conclusion, we obtain the commuting diagram

$$\begin{tikzcd} W_X \arrow[r, "\pi^{'}"] \arrow[d, "\pi"] & B \arrow[d, "\beta"] \\ B^{'} \arrow[r, "\beta^{'}"] & \mathbb{P}^1 \end{tikzcd} \quad . \tag{7.20}$$

where we denote the restriction $\lambda|_{W_X}$ as $\pi^{'}$, $p^{'}|_{W_X} = \pi$ and $p|_{W_B} = \beta$. But, generically, $W_X \cong X$. Therefore, we have proven that each generic elliptic Calabi-Yau threefold X fibered over a rational elliptic surface $B^{'}$ is a fiber product $B \times_{\mathbb{P}^1} B^{'}$ of two rational elliptic surfaces B and $B^{'}$.

We now compute the Chern classes for the tangent bundle $T_{B \times_{\mathbb{P}^1} B^{'}}$ of $B \times_{\mathbb{P}^1} B^{'}$. The results of this calculation will be used to prove that $B \times_{\mathbb{P}^1} B^{'}$ is indeed a Calabi-Yau threefold. We begin by considering the exact sequence

$$0 \longrightarrow T_D \longrightarrow T_X|_D \longrightarrow N_D \longrightarrow 0\ , \tag{7.21}$$

where X is a complex manifold, $D \subset X$ is a divisor and T and N denote the tangent and normal bundles respectively. It is straightforward to see that $B \times_{\mathbb{P}^1} B^{'}$ is a divisor of $B \times B^{'}$. For these manifolds, (7.21) becomes

$$0 \longrightarrow T_{B\times_{\mathbb{P}^1}B'} \longrightarrow T_{B\times B'}|_{B\times_{\mathbb{P}^1}B'} \longrightarrow N_{B\times_{\mathbb{P}^1}B'} \longrightarrow 0 \ . \tag{7.22}$$

Since $B \times_{\mathbb{P}^1} B^{'}$ is a divisor, it is the zero locus of a section of the line bundle $\mathcal{O}_{B\times B'}(B \times_{\mathbb{P}^1} B^{'})$. Its normal bundle is given by

$$N_{B\times_{\mathbb{P}^1}B'} = \mathcal{O}_{B\times B'}\left(B \times_{\mathbb{P}^1} B^{'}\right)|_{B\times_{\mathbb{P}^1}B'} \ . \tag{7.23}$$

Now consider the Chern characters. From (7.22), we see that

$$ch\left(T_{B\times_{\mathbb{P}^1}B'}\right) = \mathrm{ch}\left(T_{B\times B'}|_{B\times_{\mathbb{P}^1}B'}\right) - ch\left(N_{B\times_{\mathbb{P}^1}B'}\right). \tag{7.24}$$

Using the relation

$$T_{B\times B'} = \pi^{'*}T_B \oplus \pi^* T_{B'} \tag{7.25}$$

and the fact that for any vector bundle V over a threefold

$$\mathrm{ch}(V) = \mathrm{rank}V + c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right), \tag{7.26}$$

where c_i is the i-th Chern class, we can compute $ch(T_{B\times_{\mathbb{P}^1}B'})$ in (7.24). First, consider $ch(T_{B\times B'}|_{B\times_{\mathbb{P}^1}B'})$. It follows from (2.25) that

$$c_1(T_B) = f\,, \qquad c_1(T_{B'}) = f^{'} \tag{7.27}$$

and, hence, using (7.25)

$$c_1(T_{B\times B'}) = f \times B^{'} + B \times f^{'} \ . \tag{7.28}$$

Similarly,

$$c_2(T_{B\times B'}) = 12pt \times B^{'} + B \times 12pt^{'} \tag{7.29}$$

where we used the fact that $c_2(T_B) = \chi(B) = 12$ and similarly for $B^{'}$. Setting $V = T_{B\times B'}|_{B\times_{\mathbb{P}^1}B'}$ in (7.26), and restricting the the classes (7.28) and (7.29) to $B \times_{\mathbb{P}^1} B^{'}$, we find that

$$ch\left(T_{B\times B'}|_{B\times_{\mathbb{P}^1}B'}\right) = 4 + f \times f^{'} + f \times f^{'} - 12pt \times f^{'} + f \times 12pt^{'} \ . \tag{7.30}$$

Now, consider $ch(N_{B\times_{\mathbb{P}^1}B'})$. Using expressions (7.23), we see

$$\mathrm{ch}\left(N_{B\times_{\mathbb{P}^1}B'}\right) = ch\left(\mathcal{O}_{B\times_{\mathbb{P}^1}B'}\left(2\left(f \times f^{'}\right)\right)\right). \tag{7.31}$$

It follows from (7.26) that

$$\mathrm{ch}\left(N_{B\times_{\mathbb{P}^1}B'}\right) = 1 + 2\left(f \times f^{'}\right). \tag{7.32}$$

Plugging expressions (7.30) and (7.32) into (7.24), one obtains

$$\mathrm{ch}\left(T_{B\times_{\mathbb{P}^1}B'}\right) = 3 + 0 + \frac{1}{2}\left(0 - 2\left(12pt \times f' + f \times 12pt'\right)\right) \tag{7.33}$$

from which, using (7.26), one can read off the Chern classes of $T_{B\times_{\mathbb{P}^1}B'}$. They are given by

$$c_1\left(T_{B\times_{\mathbb{P}^1}B'}\right) = 0\,, \qquad c_2\left(T_{B\times_{\mathbb{P}^1}B'}\right) = 12\left(pt \times f' + f \times pt'\right), \qquad c_3\left(T_{B\times_{\mathbb{P}^1}B'}\right) = 0\,. \tag{7.34}$$

Hence, a necessary condition for $B \times_{\mathbb{P}^1} B'$ to be a Calabi-Yau manifold is satisfied, namely that $c_1(T_{B\times_{\mathbb{P}^1}B'}) = 0$.

Recall that the canonical bundle of $B \times_{\mathbb{P}^1} B'$ is given by

$$K_{B\times_{\mathbb{P}^1}B'} = \wedge^3 T^*_{B\times_{\mathbb{P}^1}B'}\,, \tag{7.35}$$

where $T^*_{B\times_{\mathbb{P}^1}B'}$ is the cotangent bundle. It follows from (7.34) and (7.35) that

$$c_1(K_{B\times_{\mathbb{P}^1}B'}) = 0\,. \tag{7.36}$$

This, plus the fact that $B \times_{\mathbb{P}^1} B'$ is simply connected, then implies that $K_{B\times_{\mathbb{P}^1}B'}$ is the trivial bundle. Using (7.35), we conclude that

$$\wedge^3 T^*_{B\times_{\mathbb{P}^1}B'} = \mathcal{O}_{B\times_{\mathbb{P}^1}B'} \tag{7.37}$$

and, hence, $B \times_{\mathbb{P}^1} B'$ admits a global, holomorphic three-form. Therefore,

$$X = B \times_{\mathbb{P}^1} B' \tag{7.38}$$

is an elliptically fibered Calabi-Yau threefold over base B' with projection map π.

The above remarks apply to a manifold of the form $B \times_{\mathbb{P}^1} B'$ for any rational elliptic surfaces B and B' discussed in this paper. We now restrict those surfaces to be such that they both admit a $\mathbb{Z}_2 \times \mathbb{Z}_2$ or a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ action, as presented in the previous section. In either case, B will have at least two commuting involutions $\tau_{Bi} : B \to B$ for $i = 1, 2$, where τ_{Bi} is of the form $\tau_{Bi} = t_{\xi_i} \circ \alpha_B$. The involution $\alpha_B : B \to B$ induces, through the projection map in (7.3), the involution $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ on the base space. Recall that $\tau_{\mathbb{P}^1}$ has two fixed points at 0 and ∞. Furthermore, α_B leaves the fiber $f_0 = \beta^{-1}(0)$ point-wise fixed and has four isolated fixed points e, e', e'', e''' in $f_\infty = \beta^{-1}(\infty)$. Exactly the same statements are true for B'. That is, B' admits at least two commuting involutions $\tau_{B'i} : B' \to B'$ for $i = 1, 2$, where $\tau_{B'i}$ is of the form $\tau_{B'i} = t_{\xi'_i} \circ \alpha_{B'}$. The involution $\alpha_{B'}$ induces the involution $\tau_{\mathbb{P}^{1'}} : \mathbb{P}^{1'} \to \mathbb{P}^{1'}$, which has two fixed points at $0'$ and ∞'. Again, $\alpha_{B'}$ leaves $f'_{0'} = \beta'^{-1}(0')$ point-wise fixed and has four isolated fixed points on $f'_{\infty'} = \beta'^{-1}(\infty')$.

Now, as mentioned above, to construct $B \times_{\mathbb{P}^1} B'$ in (7.4) it is necessary to make an identification

$$i : \mathbb{P}^1 \to \mathbb{P}^{1'} \tag{7.39}$$

of the bases of B and B' respectively. We, henceforth, will choose the diffeormorphism i so that it identifies the involutions $\tau_{\mathbb{P}^1}$ and $\tau_{\mathbb{P}^{1'}}$, but satisfies

$$i(0) = \infty'\,, \qquad i(\infty) = 0'\,. \tag{7.40}$$

This identification is shown in figure 12.

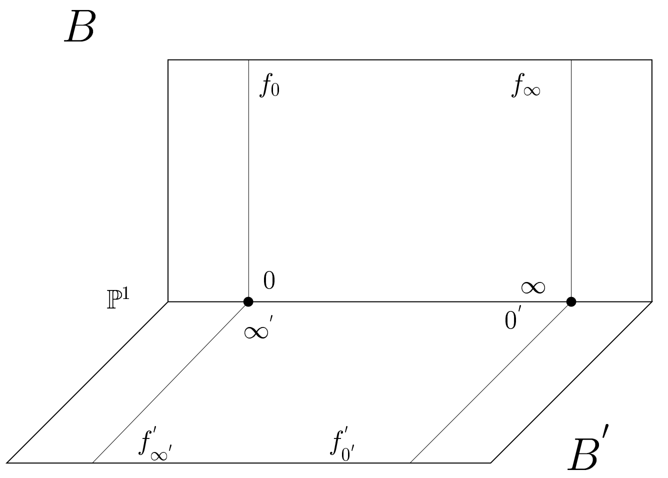

Figure 13: A diagrammatic representation of $B \times_{\mathbb{P}^1} B'$ showing the identification of the base of B with the base of B'.

It follows from the definition (7.4) and the identification (7.39), (7.40) that the involutions on B and B' naturally induce the maps $\tau_{Xi} : X \to X$ defined by

$$\tau_{Xi} = \tau_{Bi} \times_{\mathbb{P}^1} \tau_{B'i} \tag{7.41}$$

for $i = 1, 2$. Of course, maps of the form $\tau_{Bi} \times_{\mathbb{P}^1} \tau_{B'j}$ for $i \neq j$ are also induced but, as discussed below, these will not be of interest. Clearly

$$\tau_{Xi}^2 = \tau_{Bi}^2 \times_{\mathbb{P}^1} \tau_{B'i}^2 \tag{7.42}$$

and, hence, $\tau_{Xi}^2 = id$ for $i = 1, 2$. That is $\tau_{Xi}, i = 1, 2$ are involutions on X. Do these involutions commute? To answer this, note that

$$\tau_{X1} \circ \tau_{X2} = (\tau_{B1} \circ \tau_{B2}) \times_{\mathbb{P}^1} \left(\tau_{B'1} \circ \tau_{B'2}\right). \tag{7.43}$$

At this point, let us first examine the case defined in (6.7) where both B and B' admit a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. Then we see from (6.12) and (6.13) that

$$\tau_{B1} = t_\xi \circ \alpha_B\,, \qquad \tau_{B2} = t_{\xi - e_6} \circ \alpha_B \tag{7.44}$$

and

$$\tau_{B'1} = t_{\xi'} \circ \alpha_{B'}\,, \qquad \tau_{B'2} = t_{\xi' - e_6'} \circ \alpha_{B'}\,. \tag{7.45}$$

Furthermore, we showed in (6.15) that $\tau_{B1} \circ \tau_{B2} = \tau_{B2} \circ \tau_{B1}$ and $\tau_{B'1} \circ \tau_{B'2} = \tau_{B'2} \circ \tau_{B'1}$. It follows from (7.43) that

$$\tau_{X1} \circ \tau_{X2} = \tau_{X2} \circ \tau_{X1} \tag{7.46}$$

and, hence, X admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by the involutions τ_{Xi}, $i = 1, 2$.

However, unlike the case of the rational elliptic surfaces B and B', it is now essential to prove that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on manifold X is fixed point free. To show that this is indeed the case, recall that α_B leaves the fiber $f_0 = \beta(0)$ point-wise fixed. Since t_ξ is a pure translation by $\xi \neq e$, it follows the $\tau_{B1} = t_\xi \circ \alpha_B$ acts without fixed points on f_0. However, α_B leaves four isolated fixed points e, e', e'', e''' on $f_\infty = \beta^{-1}(\infty)$. That is, α_B acts like $(-1)_B$ on f_∞. Pick one of these fixed points, say e'. Let $\tilde{\xi}$ be the point in f_∞ with the property that $\tilde{\xi} + \tilde{\xi} = \xi|_{f_\infty}$. Then

$$t_\xi \circ \alpha_B \left(e' + \tilde{\xi}\right) = t_\xi \left(-e' - \tilde{\xi}\right) = e' + \tilde{\xi}, \tag{7.47}$$

where we have used the fact that $-e' = e'$. It follows that

$$\tau_{B1} \left(e' + \tilde{\xi}\right) = e' + \tilde{\xi} \tag{7.48}$$

and, hence, $e^{'}+\tilde{\xi}$ in f_∞ is an isolated fixed point of τ_{B1}. Clearly, $\tilde{\xi}$, $e^{''}+\tilde{\xi}$ and $e^{'''}+\tilde{\xi}$ in f_∞ are also isolated fixed points of τ_{B1}. Note that since α_B exchanges all other fibers pairwise, these are the only fixed points of τ_{B1}. Clearly, $\tau_{B1'}$ has exactly the same properties. That is, the only fixed points of $\tau_{B1'}$, are the four isolated points $\tilde{\xi}^{'}, (e^{'})^{'}+\tilde{\xi}^{'}, (e^{''})^{'}+\tilde{\xi}^{'}$ and $(e^{'''})^{'}+\tilde{\xi}^{'}$ in $f^{'}_{\infty'}$. Now consider $\tau_{X1}=\tau_{B1}\times_{\mathbb{P}^1}\tau_{B'1}$. Here, the wisdom of identifying the bases $\mathbb{P}^1$ of B and $\mathbb{P}^{1'}$ of $B^{'}$ with the "twist" given in (7.40) becomes apparent. Note that, with this identification, when τ_{B1} is acting on f_∞ with four fixed points, $\tau_{B'1}$ is acting on $f^{'}_{0'}$, with none. Similarly, when $\tau_{B'1}$ is acting on $f^{'}_{\infty'}$ with four fixed points, τ_{B1}, is acting on f_0 with none. Hence, τ_{X1} acts without fixed points on X. It is straightforward to show, in exactly the same way, that τ_{X2} acts without fixed points on X. However, we are not yet finished. To show that the $\mathbb{Z}_2\times\mathbb{Z}_2$ action is fixed point free, we must also demonstrate that $\tau_{X1}\circ\tau_{X2}$ does not have invariant points. First note, using (7.43), (7.43), (7.45) and (3.14), that $\tau_{X1}\circ\tau_{X2}$ is the pure translation

$$\tau_{X1}\circ\tau_{X2}=t_{e_6}\times_{\mathbb{P}^1}t_{e^{'}_6}\,. \tag{7.49}$$

Therefore, acting on non-singular fibers, this action is fixed point free. However, t_{e_6} and $t_{e^{'}_6}$ can have fixed points on singular fibers, specifically on each of the four I_1 and four I_2 fibers on B and $B^{'}$. Happily, under a generic identification (7.39), none of the singular fibers of B is paired with a singular fiber of $B^{'}$. If follows that $\tau_{X1}\circ\tau_{X2}$ and, therefore, the $\mathbb{Z}_2\times\mathbb{Z}_2$ action on X generated by τ_{X1} and τ_{X2} is fixed point free, as required.

Finally, recall that in addition to τ_{X1} and τ_{X2} defined in (7.41), maps of the form

$$\tau_{X3}=\tau_{B1}\times_{\mathbb{P}^1}\tau_{B'2}\,, \qquad \tau_{X4}=\tau_{B2}\times_{\mathbb{P}^1}\tau_{B'1} \tag{7.50}$$

are also induced on X. These are clearly involutions and they also commute with each other and with τ_{X1} and τ_{X2}. It would appear, therefore, that we actually have a $(\mathbb{Z}_2)^4$ action on X. However, as we now demonstrate, this action is not fixed point free. It suffices to consider $\tau_{X1}\circ\tau_{X3}$. Using (7.43), (7.44), (7.45) and (3.14) we see

$$\tau_{X1}\circ\tau_{X3}=id\times_{\mathbb{P}^1}t_{e^{'}_6} \tag{7.51}$$

which has fixed points $(b,b^{'})$ whenever $b^{'}$ is a fixed point of an I_1 or I_2 fiber in $B^{'}$. Similarly, $\tau_{X2}\circ\tau_{X4}$ has fixed points. We conclude, that only the $\mathbb{Z}_2\times\mathbb{Z}_2$ action generated by τ_{X1} and τ_{X2} (or equivalently, the $\mathbb{Z}_2\times\mathbb{Z}_2$ action generated by τ_{X3} and τ_{X4}) acts freely on X. The larger group $(\mathbb{Z}_2)^4$ generated by τ_{X1}, τ_{X2}, τ_{X3} and τ_{X4} acts with fixed points.

We now turn to the case of $X=B\times_{\mathbb{P}^1}B^{'}$ where both B and $B^{'}$ are defined by (6.22) and admit a $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ action. The situation is almost identical to the case just discussed, so we will simply present the result. Recall from (6.27) that

$$\tau_{B1}=t_\xi\circ\alpha_{B'}\,, \qquad \tau_{B2}=t_{\xi-e_4}\circ\alpha_{B'}\,, \qquad \tau_{B3}=t_{\xi-e_6}\circ\alpha_B \tag{7.52}$$

and

$$\tau_{B'1}=t_{\xi'}\circ\alpha_{B'}\,, \qquad \tau_{B'2}=t_{\xi'-e^{'}_4}\circ\alpha_{B'}\,, \qquad \tau_{B'3}=t_{\xi'-e^{'}_6}\circ\alpha_{B'}\,. \tag{7.53}$$

These naturally induce the maps $\tau_{X_i} : X \to X$ for $i = 1, 2, 3$ defined by

$$\tau_{X1} = \tau_{B1} \times_{\mathbb{P}^1} \tau_{B'1} \qquad \tau_{X2} = \tau_{B3} \times_{\mathbb{P}^1} \tau_{B'3} \tag{7.54}$$

and

$$\tau_{X3} = \tau_{B2} \times_{\mathbb{P}^1} \tau_{B'2} \,. \tag{7.55}$$

Clearly all three of these mappings are involutions on X and it follows from (6.29) that they mutually commute. Hence, τ_{X_i} for $i = 1, 2, 3$ provide a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ action on X. However, this action is not entirely fixed point free. Proceeding exactly as above, it is clear that the action of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by, say, τ_{X_1} and τ_{X_2} is indeed fixed point free. Note, however, that

$$\tau_{X1} \circ \tau_{X2} \circ \tau_{X3} = t_{\xi+e_4+e_6} \circ \alpha_B \times_{\mathbb{P}^1} t_{\xi'+e'_4+e'_6} \circ \alpha_{B'} \,. \tag{7.56}$$

But recall that $(\xi + e_4 + e_6)|_{f_0} = e|_{f_0}$. Clearly, $\tau_{X1} \circ \tau_{X2} \circ \tau_{X3}$ has many fixed points. We conclude that even for surfaces with involution $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the fixed point free action induced on X is only $\mathbb{Z}_2 \times \mathbb{Z}_2$.

As in the previous case, there are several different $\mathbb{Z}_2 \times \mathbb{Z}_2$ actions that one could construct, for example, generated by $\tau_{B1} \times_{\mathbb{P}^1} \tau_{B'2}$ and $\tau_{B2} \times_{\mathbb{P}^1} \tau_{B'1}$ rather then the τ_{X1}, τ_{X2} defined in (7.54). However, in all cases, the maximal fixed point free group action on X in $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We are now, at long last, in a position to construct torus-fibered Calabi-Yau threefolds with $\mathbb{Z}_2 \times \mathbb{Z}_2$ homotopy. Let X be either of the two types of Calabi-Yau threefolds discussed in this section admitting a fixed point free $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. Denote the generators of this action by τ_{X1} and τ_{X2}. Now consider the quotient space

$$Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2) \,. \tag{7.57}$$

Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts freely on X, Z is a smooth threefold. In addition, the first Chern class of Z is given by

$$c_1(Z) = \frac{1}{4} c_1(X) \,, \tag{7.58}$$

since $rank\ (\mathbb{Z}_2 \times \mathbb{Z}_2) = 4$. Then, $c_1(X) = 0$ implies that

$$c_1(Z) = 0 \,, \tag{7.59}$$

a necessary condition for Z to be a Calabi-Yau threefold. It remains to show that Z admits a holomorphic three-form. To see this, note that τ_{Xi}, $i = 1, 2$ act on $H^0(X, \Omega^3_X)$ as multiplication by elements g_i in $\mathbb{C}^* = \mathbb{C} \setminus 0$. Since τ_{Xi} are involutions, it follows that each $g_i = \pm 1$. It is straightforward to show that, in fact, $g_i = +1$ for both $i = 1, 2$. We refer the reader to [17] for a proof. That is, both τ_{X1} and τ_{X2} leave the global holomorphic three-form on X, ω_X, invariant. It follows that, under the quotient by $\mathbb{Z}_2 \times \mathbb{Z}_2$, this descends to a global holomorphic three-form ω_Z on Z. Property (7.59) and the existence of ω_Z then guarantee that Z is a Calabi-Yau threefold. However, it can be shown that the global sections of X do not descend to sections on Z. In fact, Z has no global sections. Therefore

Z is only torus fibered and not elliptically fibered. Since it has been constructed as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient of X, Z has the non-trivial first homotopy group

$$\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2\,, \tag{7.60}$$

as desired.

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